We must have a π. Stress cannot exist in the presence of π. – David Mamet, misquoted

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Problem 1 (Putnam 1968. A1). *Prove* $\dfrac{22}{7} - \pi = \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$.

Problem 2 (Putnam 1995. B1). *For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A* partition *of a set S is a collection of disjoint subsets whose union is S.]*

Problem 3 (Putnam 1986. A3). *Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.*

Problem 4 (Putnam 2020. A3). *Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether $\sum_{n=1}^{\infty} a_n^2$ converges.*

Problem 5 (Putnam 2015. A3). *Compute* $\log_2\left(\displaystyle\prod_{a=1}^{2015}\prod_{b=1}^{2015}(1+e^{2\pi iab/2015})\right)$ *where $i^2 = -1$.*

Problem 6 (Putnam 1985. B5). *Evaluate* $\displaystyle\int_0^{\infty} t^{-1/2}e^{-1985(t+t^{-1})}\,dt$. *You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.*

Hints:

1. Do the polynomial long division and compute the integral.
2. Note that because $1+2+3+4>9$, the function π takes at most 3 different values.
3. What is $\cot[\operatorname{Arccot} a + \operatorname{Arccot} b]$? Use this to find the partial sums of the series.
4. Clearly $a_n \to 0$, so by the Taylor approximation to sine, $a_{n+1} > a_n - (a_n)^3/6$. Try comparing a_n to $1/\sqrt{n}$.
5. Show that $\prod_{b=1}^{n}(1+e^{2\pi iab/n}) = 2^{\gcd(a,n)}$ for odd n by counting how many times $e^{2\pi iab/n}$ equals each $n/\gcd(a,n)$th root of unity.
6. Substitute $t = u^2$ and $t = u^{-2}$ in the given integral, and $x = \sqrt{1985}(y + 1/y)$ in the Gaussian integral.