I've said it before: equations are the devil's sentences. The worst one is that quadratic equation, an infernal salad of numbers, letters, and symbols. – Stephen Colbert

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Problem 1 (Putnam 1946. A1). *Suppose that $f(x) = ax^2 + bx + c$, where a, b, c are real constants, satisfies the condition $|f(x)| \leq 1$ for $|x| \leq 1$. Prove that $|f'(x)| \leq 4$ for $|x| \leq 1$.*

Problem 2 (Putnam 1967. B2). *Let $0 \leq p \leq 1$ and $0 \leq r \leq 1$ and consider the identity $(px + (1-p)y)(rx + (1-r)y) = ax^2 + bxy + cy^2$. Show that $\max(a, b, c) \geq 4/9$.*

Problem 3 (Putnam 1970. A2). *Consider the locus given by the real polynomial equation $Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 = 0$ where $B^2 - 4AC < 0$. Prove there exists $\delta > 0$ such that there are no points of the locus in the punctured disk $0 < x^2 + y^2 < \delta^2$.*

Problem 4 (Putnam 1971. A4). *Show that for $0 < \epsilon < 1$ the expression $(x + y)^n(x^2 - (2 - \epsilon)xy + y^2)$ is a polynomial with positive coefficients for n sufficiently large and integral. For $\epsilon = 0.002$ find the smallest admissible value of n.*

Problem 5 (Putnam 1973. B3). *Consider an integer $p > 1$ with the property that the polynomial $x^2 - x + p$ takes prime values for all integers x in the range $0 \leq x < p$. (Examples: $p = 5$ and $p = 41$ have this property.) Show that there is exactly one triple of integers a, b, c satisfying the conditions $b^2 - 4ac = 1 - 4p$, $0 < a \leq c$, $-a \leq b < a$.*

Problem 6 (Putnam 1980. A1). *Let b and c be fixed real numbers and let the ten points (j, y_j) for $j = 1, 2, \ldots, 10$ lie on the parabola $y = x^2 + bx + c$. For $j = 1, 2, \ldots, 9$ let I_j be the point of intersection of the tangents to the parabola at (j, y_j) and $(j + 1, y_{j+1}$. Determine the polynomial function $y = g(x)$ of least degree whose graph passes through all nine points I_j.*

Hints:

1. Since $f'(x)$ is linear, its maximum occurs at $x = \pm 1$. Compute $f'(\pm 1)$ in terms of the three values $f(1)$, $f(0)$, $f(-1)$.

2. Draw the regions where pr, $(1-p)r + p(1-r)$, $(1-p)(1-r)$ are all less than $4/9$ inside $[0,1] \times [0,1]$.

3. Use the second derivatives test on the given function at $(0,0)$.

4. The coefficient of $x^{k+1}y^{n+1-k}$ in the product is $\binom{n+2}{k} - (2-\epsilon)\binom{n+2}{k+1} + \binom{n+2}{k+2}$. Divide this by $\binom{n+2}{k+1}$, then minimize it.

5. Note that by symmetry the polynomial has the prime property for $1 - p < x \leq p$. Show $(b+1)/2$ is an integer in that range and set $x = (b+1)/2$.

6. Write down the equations of the two tangent lines and find their intersection.