The Department of Mathematics prepares students not only to solve for (x), but to see the whole equation. – Northeastern College of Science

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Problem 1 (Putnam 2000. B4)**.** *Let $f(x)$ be a continuous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.*

Problem 2 (Putnam 2001. A3)**.** *For each integer m, consider the polynomial $P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2$. For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?*

Problem 3 (Putnam 2001. B2)**.** *Find all pairs of real numbers (x, y) satisfying the system*

$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).$$

Problem 4 (Putnam 2003. B1)**.** *Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that $1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$ holds identically?*

Problem 5 (Putnam 2005. B3)**.** *Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that $f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$ for all $x > 0$.*

Problem 6 (Putnam 2008. A1)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.*

Hints:

1. Let $x = \cos t$. Then $f(\cos 2t) = 2\cos t f(\cos t)$. What does this say about the function $g(t) = f(\cos t)/\sin t$?

2. If P_m factors then it has a factor of degree 1 or 2. But $P_m(x) = P_m(-x)$ so linear factors come in pairs: thus, $P_m(x) = (x^2 - a)(x^2 - b)$ or $P_m(x) = (x^2 + ax + b)(x^2 - ax + b)$.

3. Add and subtract the equations, and clear denominators.

4. Suppose yes, and set $x = 0$, $x = 1$, and $x = 2$.

5. We have $f(x)f'(a/x) = x$ and by symmetry $f(a/x)f'(x) = a/x$. Do terms like that arise from some kind of derivative rule?

6. What happens if you set $z = 0$ in the first equation?