Don't rely on getting easy questions. Practice for the hard ones. – A. T. Hincks

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Problem 1 (Putnam 1942. A1). *A square of side $2a$, lying always in the first quadrant of the xy-plane, moves so that two consecutive vertices are always on the x- and y-axes respectively. Find the locus of the center of the square.*

Problem 2 (Putnam 1963. B3). *Find every twice-differentiable real-valued function f with domain the set of all real numbers and satisfying the functional equation $f(x)^2 - f(y)^2 = f(x+y)f(x-y)$ for all real x and y.*

Problem 3 (Putnam 1972. B1). *Show that the power series representation for the series $\sum_{n=0}^{\infty}(x^n(x-1)^{2n})/n!$ cannot have three consecutive zero coefficients.*

Problem 4 (Putnam 1986. A2). *What is the units (i.e., rightmost) digit of $\left\lfloor \dfrac{10^{20000}}{10^{100}+3} \right\rfloor$?*

Problem 5 (Putnam 1997. B2). *Let f be a twice-differentiable real-valued function satisfying $f(x) + f''(x) = -xg(x)f'(x)$, where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.*

Problem 6 (Putnam 2022. A1). *Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.*

Hints:

1. Two vertices are at $(\cos t, 0)$ and $(0, \sin t)$. Find the positions of the other two vertices.

2. Differentiate the given relation with respect to x and then with respect to y.

3. The series is $f(x) = e^{p(x)}$ for $p(x) = x(x-1)^2$. Show that f satisfies a differential equation of order 3.

4. You want the quotient when dividing 10^{20000} by $10^{100} + 3$. You can find the remainder by considering 10^{20000} modulo $10^{100} + 3$.

5. Try multiplying both sides by $f'(x)$. Is the left-hand side the derivative of something?

6. Check $a = 0$, then wlog fix $a > 0$ and show there is always at least one intersection point. For fixed a, think about the behavior of the line as b varies.