What good is the warmth of summer, without the cold of winter to give it sweetness?
– John Steinbeck

Problem 1 (Putnam 1952. A1). *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree n with integral coefficients. If a_0, a_n, and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.*

Problem 2 (Putnam 1960. A2). *Show that if three points are inside a closed square of unit side, then some pair of them are within $\sqrt{6} - \sqrt{2}$ units apart.*

Problem 3 (Putnam 1968. A2). *Given integers a, b, c, d, e, f with $ad \neq bc$, and given a real number $\epsilon > 0$, show there exist rational numbers r and s for which $0 < |ra + sb - e| < \epsilon$ and $0 < |rc + sd - f| < \epsilon$.*

Problem 4 (Putnam 1984. B1). *Let n be a positive integer and $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that $f(n+2) = P(n)f(n+1) + Q(n)f(n)$.*

Problem 5 (Putnam 1987. A1). *Curves A, B, C and D are defined in the plane as follows:*

$$A = \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \qquad B = \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\},$$
$$C = \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \qquad D = \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$.

Problem 6 (Putnam 2016. B4). *Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.*

Hints:

1. Suppose $x = p/q$ is a rational root. Clear denominators of $f(p/q) = 0$ and break into cases based on whether p, q are even or odd.

2. By sliding the points, you can assume one of them is a vertex of the square. Draw the region where the other two points could be.

3. Choose rational x, y in $(0, \epsilon)$ and solve $ra + sb - e = x$, $rc + sd - f = y$.

4. Note that $f(n+1) - f(n) = (n+1)!$. Find a relation between $f(n+2) - f(n+1)$ and $f(n+1) - f(n)$.

5. Consider the equations $A + Bi = 0$ and $C + Di = 0$.

6. Decompose the determinant as a sum over $n!$ possible permutations.