When you are equal with another person, the problem of integration doesn't even arise.
– Malcolm X

Problem 1 (Putnam 1973. B4). *On* $[0,1]$ *let* f *have a continuous derivative satisfying* $0 < f'(x) \leq 1$, *and suppose* $f(0) = 0$. *Prove that* $\left[\int_0^1 f(x)\,dx\right]^2 \geq \int_0^1 f(x)^3\,dx$.

Problem 2 (Putnam 1978. A3). *Let* $p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6$. *For* k *with* $0 < k < 5$, *define* $I_k = \int_0^\infty \frac{x^k}{p(x)}\,dx$. *For which* k *is* I_k *smallest?*

Problem 3 (Putnam 1980. A3). *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.

Problem 4 (Putnam 1982. A3). *Evaluate* $\displaystyle\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\,dx$.

Problem 5 (Putnam 1993. A5). *Show that the given sum is a rational number:*
$\displaystyle\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$.

Problem 6 (Putnam 2005. A5). *Evaluate* $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

Hints:

1. Let $a(t) = [\int_0^t f(x)\,dx]^2 - \int_0^t f(x)^3\,dx$ and show that $a'(t)/f(t)$ is increasing hence nonnegative.
2. Show $I_k = I_{4-k}$ using a substitution. Then find k minimizing $x^k + x^{4-k}$.
3. Try substituting $x \mapsto \pi/2 - x$.
4. Try writing this integral as a double integral: what integrand and what limits for y would yield the given integrand in x?
5. Try finding substitutions to convert the integration ranges for the second and third integrals to the range of the first one.
6. Let $f(a) = \int_0^1 \frac{\ln(ax+1)}{x^2+1}\,dx$ and compute $f'(a)$.