The whole is more than the sum of its parts. – Aristotle (being wrong as usual)

Problem 1 (Putnam 1978. A1)**.** *Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ... , 100. Prove that there must be two distinct integers in A whose sum is 104.*

Problem 2 (Putnam 1966. A1)**.** *Let $f(n)$ be the sum of the first n terms of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ where the nth term is given by $a_n = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$. Show that if x and y are positive integers and $x > y$ then $xy = f(x+y) - f(x-y)$.*

Problem 3 (Putnam 1999. A4)**.** *Sum the series* $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}$.

Problem 4 (Putnam 1981. A1)**.** *Let $E(n)$ denote the largest integer k such that 5^k is an integral divisor of the product $1^1 2^2 3^3 \cdots n^n$. Calculate* $\displaystyle\lim_{n\to\infty} \frac{E(n)}{n^2}$.

Problem 5 (Putnam 1969. B2)**.** *Show that a finite group can not be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?*

Problem 6 (Putnam 2017. B2)**.** *Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that (i) B is topologically closed, and (ii) B contains exactly one point from each pair of diametrically opposite points on the circle? (A set B is topologically closed when it contains the limit of every convergent sequence of points in B.)*

Hints:

1. Which pairs have sum 104? How many pairs are there?

2. One option is to find a formula for $f(n)$ for n even and n odd. Another is to think about what terms appear in the sum for $f(x+y)$ that are removed from $f(x-y)$: how many are there and what are they?

3. The sum isn't symmetric in m and n. Try adding another copy of the sum with m, n swapped to make it symmetric.

4. Tabulate $E(n)$ by counting contributions from multiples of 5, multiples of 25, multiples of 125, etc. in $[1, n]$.

5. How many elements can be in a proper subgroup of G?

6. Think about the set $-B$ consisting of the points opposite those in B. Is it closed? Open?