April's air stirs in / Willow-leaves... A butterfly / Floats and balances – Bashō

Problem 1 (Putnam 1941. A1)**.** *Prove that the polynomial* $(a-x)^6 - 3a(a-x)^5 + \frac{5}{2}a^2(a-x)^4 - \frac{1}{2}a^4(a-x)^2$ *takes only negative values for* $0 < x < a$.

Problem 2 (Putnam 1968. B5)**.** *Let* p *be a prime number. Let* J *be the set of all* 2×2 *matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *whose entries are chosen from* $\{0, 1, 2, \ldots, p-1\}$ *and satisfy the conditions* $a + d \equiv 1$ *(mod* p*),* $ad - bc \equiv 0$ *(mod* p*). Determine how many members* J *has.*

Problem 3 (Putnam 1971. A1)**.** *Let there be given nine lattice points (points with integral coordinates) in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.*

Problem 4 (Putnam 1982. B4)**.** *Let* $n_1, n_2, \ldots, n_s$ *be distinct integers such that* $(n_1 + k)(n_2 + k) \cdots (n_s + k)$ *is a multiple of* $n_1 n_2 \cdots n_s$ *for every integer* k*. Prove or disprove (i)* $|n_i| = 1$ *for some* i*, (ii) if all* n_i *are positive then* $\{n_1, n_2, \ldots, n_s\} = \{1, 2, \ldots, s\}$.

Problem 5 (Putnam 2009. A1)**.** *Let* f *be a real-valued function on the plane such that for every square* $ABCD$ *in the plane,* $f(A) + f(B) + f(C) + f(D) = 0$*. Does it follow that* $f(P) = 0$ *for all points* P *in the plane?*

Problem 6 (Putnam 2022. B2)**.** *Let* $\times$ *represent the cross product in* $\mathbb{R}^3$*. For what positive integers* n *does there exist a set* $S \subset \mathbb{R}^3$ *with exactly* n *elements such that* $S = \{v \times w : v, w \in S\}$*?*

Hints:

1. Let $y = (a - x)/a$ and rewrite in terms of y.

2. For each fixed value of a, how many possible pairs (b, c) are there?

3. Two lattice points have a lattice midpoint if and only if their coordinates are all the same parity.

4. Let $f(k)$ be the given function. For (i), what does $f(0)^2|f(-1)f(1)$ say? For (ii) consider the sth difference $\Delta^s f(k)$, where $\Delta f(k) = f(k+1) - f(k)$.

5. Try drawing a 3×3 grid and seeing what relations hold among f on the 9 entries in the grid.

6. If $\mathbf{v}, \mathbf{w}$ are in S, let $\mathbf{x} = \mathbf{v} \times \mathbf{w}$ and consider $\mathbf{v} \times \mathbf{x}$, $\mathbf{v} \times (\mathbf{v} \times \mathbf{x})$, $\mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times \mathbf{x}))$, etc.