You have to make every group you come in contact with better. – Will Smith

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Problem 1 (Putnam 1968. B2). *Given a subset A of a finite multiplicative group G such that A contains more than one-half of the elements of G, prove that each element of G is the product of two elements of A.*

Problem 2 (Putnam 1985. B6). *Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.*

Problem 3 (Putnam 1989. B2). *Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?*

Problem 4 (Putnam 2001. A1). *Consider a set S and a binary operation $*$ (for each $a, b \in S$, $a * b \in S$). If $(a * b) * a = b$ for all $a, b \in S$, prove that $a * (b * a) = b$ for all $a, b \in S$.*

Problem 5 (Putnam 2007. A5). *Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.*

Problem 6 (Putnam 2016. A5). *Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form $g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$ with $1 \leq r \leq |G|$ and $m_1, n_1, \ldots, m_r, n_r \in \{-1, 1\}$.*

Hints:

1. For $g \in G$ consider the sets gA^{-1} and A.
2. Explain why $SM_i = M_i$, then sum over i.
3. Use the powers of a to identify the element that would be the identity, then show it is an identity.
4. Try plugging in different possible simple expressions for a, such as $a * b$ or $b * a$.
5. Consider the p-tuples $(g_1, g_2, \ldots, g_p)$ whose product is the identity.
6. Consider the directed graph whose vertices are the elements of G and with edges directed from x to each of $ghx, g^{-1}hx, gh^{-1}x, g^{-1}h^{-1}x$. Show this graph is connected.