Despite the forecast, live like it's spring! – Lilly Pulitzer

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Problem 1 (Putnam 1959. B3). *If a square of unit side be partitioned into two sets, show that the diameter (least upper bound of the distances between pairs of points) of one of the sets is at least* $\sqrt{5}/2$*, and that* $\sqrt{5}/2$ *is the largest such value.*

Problem 2 (Putnam 1974. B2). *Let* $y(x)$ *be a continuously differentiable real-valued function of a real variable* x*. Show that if* $(y')^2 + y^3 \to 0$ *as* $x \to +\infty$*, then* $y(x)$ *and* $y'(x) \to 0$ *as* $x \to +\infty$*.*

Problem 3 (Putnam 1978. B1). *Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units.*

Problem 4 (Putnam 1983. A3). *Let* p *be an odd prime and let* $F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}$*. Prove that if* a *and* b *are distinct integers in* $\{0, 1, 2, \ldots, p-1\}$ *then* $F(a)$ *and* $F(b)$ *are not congruent modulo* p*.*

Problem 5 (Putnam 1987. B3). *Let* F *be a field in which* $1 + 1 \neq 0$*. Show that the set of solutions to the equation* $x^2 + y^2 = 1$ *with* x *and* y *in* F *is given by* $(x, y) = (1, 0)$ *and* $(x, y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$ *where* r *runs through the elements of* F *such that* $r^2 \neq -1$*.*

Problem 6 (Putnam 2009. A3). *Let* d_n *be the determinant of the* $n \times n$ *matrix whose entries, from left to right and then top to bottom, are* $\cos 1, \cos 2, \ldots, \cos n^2$*. (For example,* $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$*. The argument of* cos *is in radians.) Evaluate* $\lim_{n\to\infty} d_n$*.*

Hints:

1. Consider the four vertices and the four midpoints of the sides of the square.

2. If y' has zeroes that tend to $+\infty$, then the value of y goes to zero on these points: why does this imply y goes to zero in between? Otherwise, y' is eventually always positive or always negative.

3. Rearranging the sides so they alternate in length $3,2,3,2,3,2,3,2$ does not change the area, and then all its angles are equal.

4. Note that $(1-n)^2F(n) = 1 - pn^{p-1} + (p-1)n^p$.

5. The line through $(1,0)$ and $P \neq (1,0)$ on the circle must have a rational slope. So consider the line passing through $(1,0)$ with slope r: where does it intersect the circle?

6. Try using $\cos(n+a) = \cos a \cos n - \sin a \sin n$ and $\cos(n+2a) = \cos 2a \cos n - \sin 2a \sin n$ to find a linear relation between the first few rows.