May the New Year bring you courage to break your resolutions early! – A. Crowley

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Problem 1 (Putnam 1983. A1). *How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40}, 20^{30}?*

Problem 2 (Putnam 1999. A3). *Consider the power series $\dfrac{1}{1-2x-x^2} = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.*

Problem 3 (Putnam 2000. A4). *Prove that $\displaystyle\int_0^{\infty} \sin(x)\sin(x^2)\,dx$ converges.*

Problem 4 (Putnam 2011. A4). *For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?*

Problem 5 (Putnam 2017. B2). *Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers $N = a + (a+1) + (a+2) + \cdots + (a+k-1)$ for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?*

Problem 6 (Putnam 2004. B2). *Let m and n be positive integers. Show that $\dfrac{(m+n)!}{(m+n)^{m+n}} < \dfrac{m!}{m^m}\dfrac{n!}{n^n}$.*

Hints:

1. Note that a divisor of both integers divides their gcd.

2. Show that the a_n satisfy $a_n = 2a_{n-1} + a_{n-2}$ with $a_0 = 1$, $a_1 = 2$. From this you can either guess the value of m in terms of n, or solve the recurrence.

3. Try integrating by parts a few times: this will introduce factors of x into the denominator, eventually yielding convergent integrals.

4. Equivalently, $A^T A = M - I \mod 2$, where M is the matrix of all ones. Try working with this mod 2.

5. Note that 2017 is prime, and the sum $a + \cdots + (a + k - 1) = k(2a + k - 1)/2$ factors as half a product of two quantities of opposite parity.

6. Try rearranging the expression to find a binomial coefficient on one side, and think about the terms on the other side.