A2, Brute? – Julius Caesar

Problem 1 (Putnam 1988. A2)**.** *A not uncommon calculus mistake is to believe that the product rule for derivatives says that* $(fg)' = f'g'$*. If* $f(x) = e^{x^2}$*, determine, with proof, whether there exists an open interval* (a, b) *and a nonzero function* g *defined on* (a, b) *such that this wrong product rule is true for* x *in* (a, b)*.*

Problem 2 (Putnam 1998. A2)**.** *Let* s *be any arc of the unit circle lying entirely in the first quadrant. Let* A *be the area of the region lying below* s *and above the* x*-axis and let* B *be the area of the region lying to the right of the* y*-axis and to the left of* s*. Prove that* $A + B$ *depends only on the arc length, and not on the position, of* s*.*

Problem 3 (Putnam 2003. A2)**.** *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be nonnegative real numbers. Show that* $(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}$*.*

Problem 4 (Putnam 2007. A2)**.** *Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola* $xy = 1$ *and both branches of the hyperbola* $xy = -1$*. (A set* S *in the plane is called* convex *if for any two points in* S *the line segment connecting them is contained in* S*.)*

Problem 5 (Putnam 2016. A2)**.** *Given a positive integer* n*, let* $M(n)$ *be the largest integer* m *such that* $\binom{m}{n-1} > \binom{m-1}{n}$*. Evaluate* $\lim_{n\to\infty} \frac{M(n)}{n}$*.*

Problem 6 (Putnam 2018. A2)**.** *Let* $S_1, S_2, \ldots, S_{2^n-1}$ *be the nonempty subsets of* $\{1, 2, \ldots, n\}$ *in some order, and let* M *be the* $(2^n - 1) \times (2^n - 1)$ *matrix whose* (i, j) *entry is* $m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$ *Calculate the determinant of* M*.*

Hints:

1. Solve for g'/g.
2. Suppose the arc goes from $(\cos s, \sin s)$ to $(\cos(s+t), \sin(s+t))$.
3. Assuming all values are positive, rescale to make $a_i + b_i = 1$ for each i.
4. Suppose the intersections are $(a, 1/a)$, $(b, -1/b)$, $(-c, -1/c)$, $(-d, 1/d)$ for positive a, b, c, d.
5. Simplify the ratio $\binom{m}{n-1}/\binom{m-1}{n}$.
6. Order the subsets as [those not containing n] followed by [those containing n], with the order in each block being that from the previous matrix.