Yesterday, upon the stair // I met a man who wasn't there!
He wasn't there again today // I wish, I wish he'd go away! – W.H. Mearns

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Problem 1 (Putnam 1965. B5). *Consider collections of unordered pairs of V different objects. Three pairs such as bc, ca, ab are said to form a triangle. Prove that, if $4E \leq V^2$, it is possible to choose E pairs so that no triangle is formed.*

Problem 2 (Putnam 1967. A4). *Show that if $\lambda > 1/2$ there does not exist a real-valued function u such that for all x with $0 \leq x \leq 1$, $u(x) = 1 + \lambda \int_x^1 u(y)u(y-x)\,dy$.*

Problem 3 (Putnam 1969. B2). *Show that a finite group cannot be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?*

Problem 4 (Putnam 1975. B2). *In three-dimensional Euclidean space, define a* slab *to be the open set of points lying between two parallel planes. The distance between the planes is called the* thickness *of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$ respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.*

Problem 5 (Putnam 1987. A3). *For all real x, the real-valued function $y = f(x)$ satisfies the differential equation $y'' - 2y' + y = 2e^x$.*

1. *If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.*
2. *If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.*

Problem 6 (Putnam 2006. B1). *Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of distinct points A, B, C which are vertices of an equilateral triangle, and find its area.*

Hints:

1. Each "pair" is an edge in the complete graph on V vertices, and then a "triangle" is an actual triangle.
2. If there is such a u, try integrating both sides from $x = 0$ to $x = 1$.
3. Count elements in the subgroups, noting they both contain the identity.
4. It is a theorem of Archimedes that the surface area cut off of a sphere by any slab is proportional to the thickness of the slab.
5. By undetermined coefficients, $y = (A + Bx + Cx^2)e^x$ for some A, B, C.
6. The polynomial $a^3 + b^3 + c^3 - 3abc$ is divisible by $a + b + c$.