Take chances, make mistakes, get messy! – Ms. Frizzle (*The Magic School Bus*)

Problem 1 (Putnam 2003. A1)**.** *Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.*

Problem 2 (Putnam 1963. B1)**.** *For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?*

Problem 3 (Putnam 2010. A1)**.** *Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is* at least *3.]*

Problem 4 (Putnam 1976. B5)**.** *Evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^n$.*

Problem 5 (Putnam 1985. B2)**.** *Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and $\frac{d}{dx} f_{n+1}(x) = (n + 1) f_n(x + 1)$ for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.*

Problem 6 (Putnam 1988. A4)**.** *(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?*

(b) What if "three" is replaced by "nine"?

Hints:

1. Try it out for small n.
2. Try doing polynomial division. The quotient must have integer coefficients, so n^2-n+a divides $n^{13}+n+90$ for any $n \in \mathbb{Z}$.
3. Try it out for small n.
4. Try it out for small n. For the proof, think about $p(x+1)-p(x)$ for a polynomial p.
5. Try computing the first few terms.
6. Try drawing a circle of radius 1 for (a), and drawing a grid of squares for (b).