The Matrix is everywhere. It is all around us. Even now, in this very room. – Morpheus

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Problem 1 (Putnam 1951. A1). *Show that the determinant* $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$ *is nonnegative, for real numbers* a, b, c, d, e, f.

Problem 2 (Putnam 1959. A5). *Let* m *and* n *be integers greater than 1 and* $a_1, \ldots, a_{m+1}$ *real numbers. Prove that there exist real* $n \times n$ *matrices* $A_1, \ldots, A_m$ *such that* $\det A_j = a_j$ *for* $j = 1, \ldots, m$ *and* $\det(A_1 + \cdots + A_m) = a_{m+1}$.

Problem 3 (Putnam 1969. A2). *Let* D_n *be the determinant of the* $n \times n$ *matrix whose entry in the* i*th row and* j*th column is* $|i - j|$. *Show that* D_n *is equal to* $(-1)^{n-1}(n-1)2^{n-1}$.

Problem 4 (Putnam 1981. B4). *Let* V *be a set of* 5×7 *matrices with real entries and with the property that* $rA + sB \in V$ *whenever* $A, B \in V$ *and* r, s *are real scalars. Prove or disprove the following assertion: if* V *contains matrices of ranks 0, 1, 2, 4, and 5, then it also contains a matrix of rank 3.*

Problem 5 (Putnam 1994. A4). *Let* A *and* B *be* 2×2 *matrices with integer entries such that* $A, A + B, A + 2B, A + 3B$, *and* $A + 4B$ *are all invertible matrices whose inverses have integer entries. Show that* $A + 5B$ *is invertible and that its inverse has integer entries.*

Problem 6 (Putnam 1994. B4). *For* $n \geq 1$, *let* d_n *be the greatest common divisor of the entries of* $A^n - I$, *where* $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ *and* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Show that* $\lim_{n\to\infty} d_n = \infty$.

Hints:

1. No square is negative.
2. Try using diagonal matrices. If that doesn't quite work, add in an off-diagonal entry or two.
3. Do some row operations.
4. Try using diagonal matrices.
5. Note that $\det(A + tB)$ is a quadratic function of t.
6. Write out the first few powers of A and look for a pattern.