The only reason we like complex numbers is that we don't like real numbers. – B. Sturmfels

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Problem 1 (Putnam 1975. A4). *Let $n = 2m$ where m is an odd integer greater than 1, and set $\theta = e^{2\pi i/n}$. Express $(1-\theta)^{-1}$ explicitly as a polynomial in θ with integer coefficients.*

Problem 2 (Putnam 1985. A5). *Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \le m \le 10$ is $I_m \neq 0$?*

Problem 3 (Putnam 1986. B2). *Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, z are complex numbers satisfying the simultaneous equations $x(x-1)+2yz = y(y-1)+2zx = z(z-1)+2xy$, and list all such triples T.*

Problem 4 (Putnam 1991. B2). *Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y, $f(x+y) = f(x)f(y) - g(x)g(y)$ and $g(x+y) = f(x)g(y) + g(x)f(y)$. If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.*

Problem 5 (Putnam 2018. B2). *Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \le 1\}$.*

Problem 6 (Putnam 2005. B1). *Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)*

Hints:

1. Note that θ is a root of $x^{2m} - 1$ but not $x^m - 1$.
2. Write $\cos mx = (e^{imx} + e^{-imx})/2$.
3. If $x = y$ then the first two expressions are equal, so their difference is divisible by $x - y$.
4. Let $h(x) = f(x) + ig(x)$.
5. What is $(1 - z)f_n(z)$?
6. How far away can $\lfloor 2a \rfloor$ be from $2\lfloor a \rfloor$?