So much of life, it seems to me, is determined by pure randomness. –Sidney Poitier

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Problem 1 (Putnam 1991. A1). *A* 2×3 *rectangle has vertices as* $(0,0), (2,0), (0,3)$, *and* $(2,3)$. *It rotates* $90°$ *clockwise about the point* $(2,0)$. *It then rotates* $90°$ *clockwise about the point* $(5,0)$, *then* $90°$ *clockwise about the point* $(7,0)$, *and finally,* $90°$ *clockwise about the point* $(10,0)$. *(The side originally on the* x*-axis is now back on the* x*-axis.) Find the area of the region above the* x*-axis and below the curve traced out by the point whose initial position is (1,1).*

Problem 2 (Putnam 1997. A2). *Players* $1, 2, 3, \ldots, n$ *are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers* n *for which some player ends up with all* n *pennies.*

Problem 3 (Putnam 1998. B1). *Find the minimum value of* $\dfrac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)}$ *for* $x > 0$.

Problem 4 (Putnam 2008. B2). *Let* $F_0(x) = \ln x$. *For* $n \geq 0$ *and* $x > 0$, *let* $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. *Evaluate* $\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}$.

Problem 5 (Putnam 2010. A3). *Suppose that the function* $h : \mathbb{R}^2 \to \mathbb{R}$ *has continuous partial derivatives and satisfies the equation* $h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$ *for some constants* a, b. *Prove that if there is a constant* M *such that* $|h(x,y)| \leq M$ *for all* $(x,y) \in \mathbb{R}^2$, *then* h *is identically zero.*

Problem 6 (Putnam 2018. B6). *Let* S *be the set of sequences of length* 2018 *whose terms are in the set* $\{1,2,3,4,5,6,10\}$ *and sum to* 3860. *Prove that the cardinality of* S *is at most* $2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}$.

Hints:

1. Draw the path.
2. Try out the game for small numbers of players.
3. Write the expression in terms of $u = x + 1/x$.
4. Integrate by parts to get a recurrence relation.
5. Restrict attention to h on lines parallel to $\langle a, b \rangle$.
6. Use a generating function.