So long, farewell, auf wiedersehen, good night! – "The Sound of Music" (1965)

Problem 1 (Putnam 2004. B3). *Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region $R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$ has perimeter k units and area k square units for some real number k.*

Problem 2 (Putnam 2006. B2). *Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that $\left|m + \sum_{s \in S} s\right| \leq \frac{1}{n+1}$.*

Problem 3 (Putnam 2007. A3). *Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3?*

Problem 4 (Putnam 2007. B1). *Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.*

Problem 5 (Putnam 2009. B1). *Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?*

Problem 6 (Putnam 2018. B3). *Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n - 1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.*

Hints:

1. It's clearly possible to increase the perimeter arbitrarily without changing the area. So for a fixed area, what shape minimizes the perimeter?

2. Consider x_1, $x_1 + x_2$, $x_1 + x_2 + x_3$, $\cdots$, $x_1 + \cdots + x_n$.

3. Reduce mod 3. The zeroes can be temporarily ignored: what then are the possible arrangements of the 1s and 2s?

4. Use the fact that $a - b$ divides $f(a) - f(b)$ for an integer polynomial f.

5. Try drawing a 3×3 grid of points. What conditions are there on the values in the grid?

6. Note $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd a,b} - 1$.