I had a polynomial once. My doctor removed it. – M. Grant

Problem 1 (Putnam 1985. B1). *Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.*

Problem 2 (Putnam 1986. A1). *Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.*

Problem 3 (Putnam 1987. A2). *Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that $P(ux, uy, uz) = u^2 F(y - x, z - x)$ for all real x, y, z, u, and such that $P(1, 0, 0) = 4$, $P(0, 1, 0) = 5$, and $P(0, 0, 1) = 6$. Also let A, B, C be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$.*

Problem 4 (Putnam 1991. A3). *Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that $p(r_i) = 0$ and $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0$ for $i = 1, 2, \ldots, n - 1$, where $p'(x)$ denotes the derivative of $p(x)$.*

Problem 5 (Putnam 1992. A1). *Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers such that (i) $f(f(n)) = n$ for all integers n, (ii) $f(f(n + 2) + 2) = n$ for all integers n, and (iii) $f(0) = 1$.*

Problem 6 (Putnam 2007. B5). *Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n, $\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}$, where $\lfloor a \rfloor$ means the largest integer $\leq a$.*

Hints:

1. The x^5 coefficient is not zero. Could there be only two nonzero coefficients? Three?
2. Factor $x^4 - 13x^2 + 36$.
3. The polynomial P must be of the form $a(y-x)^2 + b(y-x)(z-x) + c(z-x)^2$: why?
4. Explain why $p(x) = a(x - r_1) \cdots (x - r_n)$, and then compute $p'(x)/p(x)$.
5. Simplify $f(f(f(n+2)) + 2))$ in two different ways.
6. Suppose $n = qk + r$ with $0 \leq r < k$, so that $\lfloor n/k \rfloor = q$.