There is no spoon. – Neo (The Matrix, 1999)

Problem 1 (Putnam 1957. A5)**.** *Given n points in the plane, show that the largest distance determined by these points cannot occur more than n times.*

Problem 2 (Putnam 1973. B2)**.** *Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every integer n.*

Problem 3 (Putnam 1983. A3)**.** *Let p be an odd prime and let $f(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}$. Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $f(a)$ and $f(b)$ are not congruent modulo p.*

Problem 4 (Putnam 1994. A1)**.** *Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Problem 5 (Putnam 2004. A3)**.** *Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that $\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$ for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)*

Problem 6 (Putnam 2010. B1)**.** *Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that $a_1^m + a_2^m + a_3^m + \cdots = m$ for every positive integer m?*

Hints:

1. Break into cases: i) some point is the endpoint of at most one longest segment, ii) some point is the endpoint of at least three longest segments, iii) every point has exactly two longest segments.
2. If $z = \cos\theta + i\sin\theta$, what is $z^n - z^{-n}$?
3. Multiply $f(n)$ by $(1-n)^2$.
4. The harmonic series has the given property. Why does it diverge?
5. Find the first few terms of the series. Is there a pattern?
6. Are there some inequalities that hold among the sums $\sum a_i^m$ for various m?