FALL 2026 PUTNAM PRACTICE
PROBLEM SET 8

Problem 1 (1989 B1)**.** A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

Problem 2 (2021 B1)**.** Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

Problem 3 (2001 A2)**.** You have a set of n biased coins. The m-th coin has probability $\frac{1}{2m+1}$ of landing heads ($m = 1, 2, \ldots, n$), and the results for each coin are independent. What is the probability that if each coin is tossed once, you get an odd number of heads?

Problem 4 (2020 B3)**.** Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Problem 5 (1989 A4)**.** If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.

Problem 6. Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

Problem 7 (1992 A6)**.** Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? It is understood that each point is independently chosen relative to a uniform distribution on the sphere.

(1) For simplicity, assume that the square has vertices $(\pm 1, \pm 1)$ and by symmetry assume that the dart lands in the triangle with vertices $(0,0), (0,1), (1,1)$. Within this triangle, find the area of the points that are closer to the origin than to the line $x = 1$.

(2) If the orientation is a fixed angle θ, find the probability that the square does not cover any vertices, and then integrate over θ.

(3) Let p_n be this probability. Find a recursion for $(2n+1)p_n$.

(4) Let $f(x)$ be the expected number of additional steps needed to get below δ, starting from x and prove that $f(x) = 1 + \frac{1}{x}\int_0^x f(x)\,dx$.

(5) Consider the binary expansion of α.

(6) Prove that $E((1-X)(2-X)(3-X)) \geq 6P(X=0)$.

(7) Let v_1, v_2, v_3, v_4 be the 4 vectors. Then with probability 1, the vectors v_1, v_2, v_3 are linearly independent and we can write $v_4 = -\alpha_1 v_1 - \alpha_2 v_2 - \alpha_3 v_3$. For which values of $\alpha_1, \alpha_2, \alpha_3$ does the origin lie in the tetrahedron?