Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.
– *The Matrix*, 1999

Problem 1 (Putnam 1954. A1)**.** *Let n be an odd integer greater than 1. Let A be an $n \times n$ symmetric matrix such that each row and column of A consists of some permutation of the integers $1, 2, \ldots, n$. Show that each of $1, 2, \ldots, n$ must appear on the main diagonal of A.*

Problem 2 (Putnam 1969. A1)**.** *Let D_n be the determinant of the $n \times n$ matrix in which the element in the ith row and jth column is $|i - j|$. Show that $D_n = (-1)^{n-1}(n-1)2^{n-2}$.*

Problem 3 (Putnam 1969. B6)**.** *Let A be a 3×2 matrix and B be a 2×3 matrix. If $AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$, show that $BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$.*

Problem 4 (Putnam 1991. A2)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?*

Problem 5 (Putnam 1994. B4)**.** *For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\lim_{n\to\infty} d_n = \infty$.*

Problem 6 (Putnam 2012. A5)**.** *Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries of $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$ are distinct.*

Hints:

1. The appearances of an integer k off the main diagonal come in pairs.
2. Use row and column operations to simplify the matrix: e.g., by subtracting each row from the row below.
3. Note that $\text{rank}(ABAB) \leq \text{rank}(BA)$. Use this to show BA is invertible. Then consider $(BA)^2$ and $(BA)^3$.
4. Try computing $(A - B)(A^2 + B^2)$.
5. Try computing A^2 and A^3, then guessing recurrences for the entries of A^n.
6. Show that M^{p^n} must be the identity, and then consider its possible Jordan canonical form.