FALL 2026 PUTNAM PRACTICE
PROBLEM SET 6

"Greetings! I am the Count. They call me the Count because I love to count things."
Count von Count

Problem 1 (Putnam 2001 B1)**.** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1,\ (k-1)n+2,\ \ldots,\ (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Problem 2 (Putnam 1985 A1)**.** Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

Problem 3 (Putnam 2002 A3)**.** Let $n \geq 2$ be an integer and let T_n be the number of nonempty subsets S of $\{1, 2, \ldots, n\}$ such that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Problem 4 (Putnam 1993, A3)**.** Let $\mathcal{P}_n$ be the set of all subsets of $\{1, 2, \ldots, n\}$.

Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \longrightarrow \{1, 2, \ldots, m\}$ satisfying $f(A \cap B) = \min\{f(A), f(B)\}$ for all $A, B \in \mathcal{P}_n$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

Problem 5 (Putnam 2005 B4)**.** For positive integers m, n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Prove that $f(m, n) = f(n, m)$.

Problem 6 (Putnam 1996 B5)**.** Given a finite string S consisting of the symbols X and O, define $\Delta(S)$ to be the number of X's minus the number of O's. A string S is called *balanced* if every substring T consisting of consecutive symbols satisfies

$$-2 \leq \Delta(T) \leq 2.$$

Find, with proof, the number of balanced strings of length n.

(1) The (i, j) entry is $(i-1)n+j$. What happens if we replace it by just $(i-1)n$? or just j?

(2) For a given integer k, which of the 3 statements $k \in A_1$, $k \in A_2$, $k \in A_3$ can be simultaneously true?

(3) For a set with integer average k, you can add or remove k and the average will not change, unless ...

(4) Let $k_i = f(\{1, 2, \ldots, n\} \setminus \{i\})$. Then f is *almost* determined by $k_1, k_2, \ldots, k_n$.

(5) Find a bijection between the sets that are counted by $f(n, m)$ and $f(m, n)$.

(6) Find a recursion.