Inflation has gotten so bad that the funny integer is now 67. – Internet Meme

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Problem 1 (Putnam 1960. A1)**.** *Let n be a given positive integer. How many solutions are there in ordered positive integer pairs (x, y) to the equation $\dfrac{xy}{x+y} = n$?*

Problem 2 (Putnam 1961. A4)**.** *Define a function f on positive integers as follows: $f(1) = 1$ and for $n > 1$, $f(n) = (-1)^k$ where k is the total number of prime factors of n (with multiplicity). For example $f(9) = 1$ and $f(20) = -1$. Show that the sum $\sum_{d|n} f(d)$ over positive divisors of n is always 0 or 1, and identify for which integers the sum is 1.*

Problem 3 (Putnam 1962. B5)**.** *Prove, for every integer $n > 1$, that*

$$\frac{3n+1}{2n+2} < \left(\frac{1}{n}\right)^n + \left(\frac{2}{n}\right)^n + \cdots + \left(\frac{n}{n}\right)^n < 2.$$

Problem 4 (Putnam 1965. A3)**.** *Prove that there are exactly three right-angled triangles whose sides are integers and whose area is numerically equal to twice the perimeter.*

Problem 5 (Putnam 1966. A4)**.** *Prove that after deleting the perfect squares from the list of positive integers, the number we find in the nth position is equal to $n + \{\sqrt{n}\}$ where $\{\sqrt{n}\}$ denotes the integer closest to $\sqrt{n}$.*

Problem 6 (Putnam 1966. B2)**.** *Prove that among any ten consecutive integers at least one is relatively prime to each of the others.*

Hints:

1. Clear denominators and add n^2 to both sides to factor.
2. Show $f(ab) = f(a)f(b)$ for relatively prime a, b and compute f on prime powers.
3. The sum in the middle is a Riemann sum for $\int_0^1 x^n\,dx$ with equally-spaced subintervals on $[0, 1]$. Try using a trapezoidal sum for it.
4. The Pythagorean triangles have sides given by $k(s^2-t^2, 2st, s^2+t^2)$ for relatively prime integers $s > t > 0$ of opposite parity and any $k > 0$.
5. Look at cases where $k^2 \le n \le k^2 + k$ and $k^2 + k + 1 \le n \le (k+1)^2$ and count the number of 'missing' squares.
6. Try considering the five odd numbers: what common prime factors could they have with another number in the range?