Kids these days don't do enough trigonometry. – Prof. Dummit

Problem 1 (Putnam 2025. A2). *Find the largest real number a and the smallest real number b such that $ax(\pi - x) \leq \sin x \leq bx(\pi - x)$ for all x in the interval $[0, \pi]$.*

Problem 2 (Putnam 2023. A1). *For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.*

Problem 3 (Putnam 2013. B2). *Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form $f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$ for which $f(x) \geq 0$ for all real x, and $a_n = 0$ whenever n is a multiple of 3. Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.*

Problem 4 (Putnam 1991. B2). *Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for all real x and y,*

$$f(x+y) = f(x)f(y) - g(x)g(y) \text{ and } g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem 5 (Putnam 2018. B4). *Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.*

Problem 6 (Putnam 2020. A6). *For a positive integer N, let f_N be the function defined by $f_N(x) = \displaystyle\sum_{n=0}^{N} \frac{N + 1/2 - n}{(N+1)(2n+1)} \sin((2n+1)x)$. Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.*

Hints:

1. Take $x \to 0+$ and $x = \pi/2$ to get candidates for a and b. Then show $\sin x/[x\pi - x)]$ is increasing on $[0, \pi/2]$, or do Taylor estimates on $\sin x - Cx(\pi - x)$ for $C = a, b$.

2. Compute $f''(0)$ explicitly using the product rule.

3. Write $(1 + c\cos 2\pi x)^2$ in the given form to find nonnegative functions. To bound $f(0)$, try setting $x = 1/3$.

4. Do the identities for f and g look familiar? Try letting $h(x) = f(x) + ig(x)$.

5. Try writing out the first several polynomials. Are they familiar? Perhaps reminiscent of certain identities?

6. Separate the fraction, differentiate, and then show f_N has its maximum at $x = \pi/2$.