September tries its best to help us forget summer. – Bernard Williams

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Problem 1 (Putnam 1958a. A4). *If $a_1, a_2, \ldots, a_n$ are complex numbers such that $|a_1| = |a_2| = \cdots = |a_n| = r \neq 0$ and if $\sigma(n, s)$ denotes the sum of all products of these n numbers taken s at a time, prove that $\left|\dfrac{\sigma(n,s)}{\sigma(n,n-s)}\right| = r^{2s-n}$ whenever $\sigma(n, n-s) \neq 0$.*

Problem 2 (Putnam 1967. B1). *Let $ABCDEF$ be a hexagon inscribed in a circle of radius R. Show that if $AB = CD = EF = r$ then the midpoints of BC, DE, FA are the vertices of an equilateral triangle.*

Problem 3 (Putnam 1971. B2). *Let $F(x)$ be a real-valued function defined for all real x except for $x = 0$ and $x = 1$ and satisfying the functional equation $F(x) + F(\frac{x-1}{x}) = 1 + x$. Find all functions $F(x)$ satisfying these conditions.*

Problem 4 (Putnam 1996. B4). *Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.*

Problem 5 (Putnam 2021. A1). *A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?*

Problem 6 (Putnam 2025. B2). *Let $f\colon [0,1] \to [0,\infty)$ be strictly increasing and continuous. Let R be the region bounded by $x = 0$, $x = 1$, $y = 0$, and $y = f(x)$. Let x_1 be the x-coordinate of the centroid of R. Let x_2 be the x-coordinate of the centroid of the solid generated by rotating R around the x-axis. Prove that $x_1 < x_2$.*

Hints:

1. Try finding a relation between $\sigma(n,s)$ and $\overline{\sigma(n,n-s)}$, using $a_i = r^2/\overline{a_i}$.

2. If O is the center of the circle, try using complex coordinates with O at the origin. The fact that $AB = r$ says something about triangle ABO: what?

3. Try plugging $(x-1)/x$ in for x and seeing what the functional equation says. Can you reuse this idea?

4. Try finding a pattern for small n. To prove it, observe that if a set with k elements is selfish, then all its elements must be greater than k; conversely, any such set with k elements is selfish.

5. Try considering how the sum of coordinates changes along each of the 12 possible hops.

6. The x-coordinate of the centroid of the region (or solid) equals the integral of x times the height (or cross-sectional area) at coordinate x. To show the inequality, consider $\int_0^1 \int_0^1 (f(x) - f(y))(x-y)\, dy\, dx$.