A zero itself is nothing, but without a zero you cannot count anything. – Dalai Lama

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Problem 1 (Putnam 1986. A1)**.** *Find, with explanation, the maximum value of* $f(x) = x^3 - 3x$ *on the set of all real numbers* x *satisfying* $x^4 + 36 \leq 13x^2$*.*

Problem 2 (Putnam 2002. A4)**.** *In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty* 3×3 *matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the* 3×3 *matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?*

Problem 3 (Putnam 1986. B4)**.** *For a positive real number* r*, let* $G(r)$ *be the minimum value of* $|r - \sqrt{m^2 + 2n^2}|$ *for all integers* m *and* n*. Prove or disprove the assertion that* $\lim_{r\to\infty} G(r)$ *exists and equals 0.*

Problem 4 (Putnam 1970. A1)**.** *Show that the power series for the function* $e^{ax}\cos bx$ *(for* $a > 0$ *and* $b > 0$*) in powers of* x *has either no zero coefficients or infinitely many zero coefficients.*

Problem 5 (Putnam 2019. B1)**.** *Denote by* $\mathbb{Z}^2$ *the set of all points* (x, y) *in the plane with integer coordinates. For each integer* $n \geq 0$*, let* P_n *be the subset of* $\mathbb{Z}^2$ *consisting of the point* $(0, 0)$ *together with all points* (x, y) *such that* $x^2 + y^2 = 2^k$ *for some integer* $k \leq n$*. Determine, as a function of* n*, the number of four-point subsets of* P_n *whose elements are the vertices of a square.*

Problem 6 (Putnam 1976. A4)**.** *Let* r *be a root of* $P(x) = x^3 + ax^2 + bx - 1 = 0$ *and let* $r + 1$ *be a root of* $y^3 + cy^2 + dy + 1 = 0$*, where* a, b, c, d *are integers. If* $P(x)$ *is irreducible over the rational numbers, express another root* s *of* $P(x) = 0$ *as a function of* r *which does not explicitly involve* a, b, c, d*.*

Hints:

1. Find the zeroes of the polynomial $x^4 - 13x^2 + 36$ and use this to identify the possible values of x.

2. Player 0 can win by making a row of zeroes or a 2×2 grid of zeroes. Try doing casework.

3. Try using the greedy algorithm to make $m^2 + 2n^2$ close to r.

4. The given function is the real part of something.

5. Draw the available points and find a recurrence relation.

6. It must be that $P(x-1) = x^3 + cx^2 + dx + 1$: what does this imply about the roots of P?