The only thing that really scares me about Halloween is running out of candy! – M. White

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Problem 1 (Putnam 1953. A1). *Prove that for every positive integer n, the sum $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$ is more than $\frac{2}{3}n\sqrt{n}$ and less than $\frac{4n+3}{6}\sqrt{n}$.*

Problem 2 (Putnam 1958. B5). *Given an infinite number of points in a plane, prove that if all the distances between them are integers then the points are all in a straight line.*

Problem 3 (Putnam 1980. A2). *Let r and s be positive integers. Derive a formula in terms of r and s for the number of ordered quadruples (a,b,c,d) of positive integers such that $3^r7^s = \operatorname{lcm}(a,b,c) = \operatorname{lcm}(a,b,d) = \operatorname{lcm}(a,c,d) = \operatorname{lcm}(b,c,d)$.*

Problem 4 (Putnam 1988. A6). *If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.*

Problem 5 (Putnam 1998. B2). *Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.*

Problem 6 (Putnam 2012. B1). *Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ where*

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x)+g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x)-g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Hints:

1. Compare the sum to a Riemann sum.

2. Suppose there are three noncollinear points A, B, C and AB has length l. By the triangle inequality, $-l < CB - CA < l$. What does the locus of C with $CB - CA = d$ look like?

3. Consider the prime factorizations of a, b, c, d. Remember that the power of a prime dividing $\text{lcm}(a, b)$ is the maximum of those dividing a and dividing b.

4. Suppose there is a linear dependence, and apply A to it.

5. Reflect carefully on the lines involved in the problem (or perhaps, across them).

6. Try using the fact that $\ln(f(x) + 1)$ and $\ln(g(x) + 1)$ is also in S. What else must be in S? (And how could you get rid of the natural log?)