Autumn is a second spring when every leaf is a flower. – Albert Camus

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Problem 1 (Putnam 2016. B1). *Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$, $x_{n+1} = \ln(e^{x_n} - x_n)$. Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.*

Problem 2 (Putnam 2014. A3). *Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute $\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$ in closed form.*

Problem 3 (Putnam 2021. A5). *Let A be the set of all integers n such that $1 \leq n \leq 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let $S(j) = \sum_{n \in A} n^j$. Determine all values of j such that $S(j)$ is a multiple of 2021.*

Problem 4 (Putnam 2010. B3). *There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving* exactly *i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?*

Problem 5 (Putnam 1984. A1). *Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points which are a distance at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial in a, b, and c.*

Problem 6 (Putnam 1976. A4). *Suppose r is a root of $p(x) = x^3 + ax^2 + bx - 1$ where $p(x)$ is irreducible over the rational numbers, and that $r + 1$ is a root of $q(y) = y^3 + cy^2 + dy + 1$, where a, b, c, d are integers. Express another root s of $p(x)$ is a function of r which does not explicitly involve a, b, c, d.*

Hints:

1. Rewrite the recurrence as $e^{x_{n+1}} = e^{x_n} - x_n$. What then is the sum $x_0 + x_1 + \cdots + x_n$?

2. Try working out the first few terms of the series. Can. you find a formula for the nth term? What does the product look like?

3. Since $2021 = 43 \cdot 47$ it's enough to work mod 43 and mod 47, both of which are prime. Working modulo p, try writing each of the $p-1$ units as a power of a primitive root g.

4. What's the maximum possible number of balls where no moves are possible? What if you have more than that number?

5. Draw the region carefully. Note that there are some curved pieces.

6. Note that $p(x)$ and $q(x-1)$ have a common root r, but $p(x)$ is irreducible. What does this imply about $q(x-1)$? What then are the roots of $q(x)$ in terms of the roots of p?