Let each new year find you a better person. – Benjamin Franklin

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Problem 1 (Putnam 1979. A1). *Find positive integers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 1979$ and the product $a_1 a_2 \cdots a_n$ is as large as possible.*

Problem 2 (Putnam 1982. A5). *Let a, b, c, d be positive integers and $r = 1 - a/b - c/d$. Given that $a + c \leq 1982$ and $r > 0$, prove that $r > 1/1983^3$.*

Problem 3 (Putnam 1990. B1). *Find all real-valued continuously differentiable functions f on the real line such that for all x, $(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990$.*

Problem 4 (Putnam 2009. B4). *Say that a polynomial with real coefficients in two variables, x, y, is* balanced *if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.*

Problem 5 (Putnam 2013. B1). *For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of $\sum_{n=1}^{2013} c(n)c(n+2)$.*

Problem 6 (Putnam 2021. A1). *A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?*

Hints:

1. Note that $2(n-2) \geq n$ for $n \geq 4$ so using integers 4 or larger isn't optimal. What's the best way to pick 1s, 2s, and 3s?

2. Assume $b \leq d$. Break into cases based on whether $d \leq 1983$, $b \geq 1983$, and $b < 1983 < d$. In the last case, consider which d minimizes r for a fixed a, b, c.

3. Try differentiating both sides.

4. The condition says that $\int_0^{2\pi} p(r\cos t, r\sin t)\, dt = 0$ for all r. This integral is a polynomial in r. Consider the monomial terms of p contributing to each coefficient.

5. Note that $c(n) = (-1)^d$ where d is the number of 1s in the binary expansion of n. Are there any nice patterns in $c(n)$? How about in $c(n)c(n+2)$?

6. Which hops move closest to the direction of $(2021, 2021)$? How many are needed to get there?