I've decided to get into shape, and the shape I've selected is a triangle. – Howie Mandel

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Problem 1 (Putnam 1975. A1). *Prove that an integer n is the sum of two triangular numbers (meaning that $n = \frac{a^2+a}{2} + \frac{b^2+b}{2}$ for integers a, b) if and only if $4n+1$ is the sum of two squares (meaning that $4n+1 = c^2+d^2$ for integers c, d).*

Problem 2 (Putnam 1979. B6). *For $k = 1, 2, \ldots, n$ let $z_k = x_k + iy_k$ where the x_k and y_k are real and $i = \sqrt{-1}$. Let r be the absolute value of the real part of $\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}$. Prove that $r \leq |x_1| + |x_2| + \cdots + |x_n|$.*

Problem 3 (Putnam 1982. B1). *Let M be the midpoint of a side BC of a general $\triangle ABC$. Using the smallest possible n, describe a method for cutting $\triangle AMB$ into n triangles which can be reassembled to form a triangle congruent to $\triangle AMC$.*

Problem 4 (Putnam 1985. A2). *Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.*

Problem 5 (Putnam 1965. B5). *Consider collections of unordered pairs of V different objects $a, b, c, \ldots$. Three pairs such as bc, ca, ab are said to form a "triangle". Prove that, if $4E \leq V^2$, it is possible to choose E pairs so that no triangle is formed.*

Problem 6 (Putnam 2014. A2). *Let A be the $n \times n$ matrix whose entry in the ith row and jth column is $1/\min(i, j)$ for $1 \leq i, j \leq n$. Compute $\det(A)$.*

Hints:

1. Try playing around with small n to look for a general formula for converting between $n = \triangle + \triangle$ and $4n + 1 = \square + \square$.

2. If $\sqrt{z_1^2 + \cdots + z_n^2} = r + si$ then $(r^2 - s^2) + (2rs)i = z_1^2 + \cdots + z_n^2$. Try showing that $r^2 > x_1^2 + \cdots + x_n^2$ leads to a contradiction.

3. Clearly $n = 1$ doesn't work in general. Can $n = 2$ work? If it did, what side lengths would you need to get once you cut apart $\triangle AMB$?

4. Each rectangle is inscribed in a triangle. So first, what is the largest possible area of a rectangle inscribed in a triangle, and how is the rectangle arranged?

5. These are (actual) triangles inside the graph on the V vertices with the pairs corresponding to E total edges. What is the largest possible triangle-free graph?

6. Try doing some row operations to make the matrix upper-triangular, starting from the bottom row.