FALL 2025 PUTNAM PRACTICE
PROBLEM SET 5

Problem 1 (Putnam 2018, **A1**)**.** Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$$

Problem 2 (Putnam 1969, **B1**)**.** The positive integer n is divisible by 24. Show that the sum of all the positive divisors of $n - 1$ (including 1 and $n - 1$) is also divisible by 24.

Problem 3 (Putnam 1970 **A3**)**.** A perfect square has length n if its last n digits (in base 10) are the same and non-zero. What is the longest possible length? What is the smallest square achieving this length?

Problem 4 (Putnam 2019 **A1**)**.** Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

Problem 5 (AKHIMO 2025)**.** Find all possible values of $\gcd(a^{2m} + 1, a^n + 1)$, where a, m, n are positive integers and n is odd.

Problem 6 (AKHIMO 2025)**.** Show that for every positive integer n there exist nonnegative integers p, q and integers $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q \geq 2$ such that

$$n = \frac{(a_1^3 - 1)(a_2^3 - 1) \cdots (a_p^3 - 1)}{(b_1^3 - 1)(b_2^3 - 1) \cdots (b_q^3 - 1)}.$$

(1) Clear the denominators in the equation and bring it into a nice form.
(2) The divisors of $n-1$ come in pairs $\{d, (n-1)/d\}$.
(3) First consider: if the last 2 digits of a perfect square are nonzero and the same, what can these digits be?
(4) The cubic polynomial factors as a product of a linear and a quadratic form.
(5) Consider a prime p that divides $a^{2m}+1, a^n+1$). What can you say about the order of a modulo p.
(6) First try $n=2$ compute factorizations of m^3-1 for small m.