An ellipsis is a giant ocean of possibilities. – Jami Attenberg

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Problem 1 (Putnam 2023. A1). *For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.*

Problem 2 (Putnam 1960. B4). *Consider the arithmetic progression $a, a+d, a+2d, \ldots$, where a and d are positive integers. For any positive integer k, prove that the progression either has no exact kth powers or infinitely many.*

Problem 3 (Putnam 1942. A3). *Is the series*
$1 + \frac{1}{2}\cdot\frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdots$ *convergent or divergent?*

Problem 4 (Putnam 1973. B1). *Let $a_1, a_2, \ldots, a_{2n+1}$ be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.*

Problem 5 (Putnam 1940. B6). *For integers $a_1, a_2, a_3, \ldots, a_n$ and k, show the determinant*

$$\begin{vmatrix} a_1^2+k & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2+k & a_2a_3 & \cdots & a_2a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2+k \end{vmatrix}$$

is divisible by k^{n-1} and find its other factor.

Problem 6 (Putnam 1949. B4). *Show that the coefficients in the series expansion*
$\frac{1}{4}[1 + x - \sqrt{1-6x+x^2}] = a_1x + a_2x^2 + a_3x^3 + \cdots$ *are positive integers.*

Hints:

1. Work out explicitly what $f_n''(0)$ is equal to, in terms of n.

2. If there is a kth power in the sequence, you can assume it is a by ignoring the terms before it. Find another kth power in the sequence.

3. Try using the Root Test: if $\lim_{n\to\infty} |a_n|^{1/n} = L$ then $\sum_{n=0}^{\infty} a_n$ converges if $L < 1$ and diverges if $L > 1$.

4. Explain why all of the integers must have the same parity (even or odd). Use this to show that if there exists a set of the integers with the claimed property, then there exists another set that is smaller.

5. What are the rank and the eigenvalues of $A - kI_n$, where I_n is the identity matrix?

6. Let y be the desired series. Then y satisfies a quadratic equation in terms of x. Substitute in the series expression for y and compare coefficients to get a recurrence for the coefficients.