To be radical is to grasp things by the root. – Karl Marx

Problem 1 (Putnam 1966. A4)**.** *Prove after deleting the perfect squares from the list of positive integers, the integer in the nth position is equal to* $n + \{\sqrt{n}\}$ *where* $\{x\}$ *denotes the integer closest to* x*.*

Problem 2 (Putnam 1970. B1)**.** *Evaluate* $\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \sqrt[n]{n^2+i^2}$*.*

Problem 3 (Putnam 1984. B2)**.** *Find the minimum value of* $(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$ *for* $0 < u < \sqrt{2}$ *and* $v > 0$*.*

Problem 4 (Putnam 1987. B1)**.** *Evaluate* $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$*.*

Problem 5 (Putnam 1994. A3)**.** *Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance* $2 - \sqrt{2}$ *apart.*

Problem 6 (Putnam 1998. A5)**.** *Let* $N = 1111\cdots11$ *be the positive integer with 1998 decimal digits, all of them 1. Find the thousandth digit after the decimal point of* $\sqrt{N}$*.*

Hints:

1. How many perfect squares are in the range $[1, n + \{\sqrt{n}\}]$? Try breaking into cases where $k \leq \sqrt{n} < k + 0.5$ and $k + 0.5 < \sqrt{n} < k + 1$ for an integer k.

2. Try taking the logarithm and writing it as a Riemann sum for something.

3. Try interpreting this quantity as the square of the distance between points on two curves. Do the curves have any symmetry you could exploit?

4. Look for a symmetry that the integrand's denominator has. What happens if you make a substitution that exploits that symmetry?

5. Look for various collections of points that are within $2 - \sqrt{2}$ of each other.

6. Note that $N = (10^{1998} - 1)/9$ so $\sqrt{N} = 10^{999}/3 \cdot \sqrt{1 - 10^{-1998}}$.