You hit home runs not by chance, but by preparation. – Roger Maris

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Problem 1 (Putnam 1958b. A2). *Let $R_1 = 1$ and $R_{n+1} = 1 + n/R_n$ for $n \geq 1$. Show that $\sqrt{n} \leq R_n \leq \sqrt{n} + 1$ for all $n \geq 1$.*

Problem 2 (Putnam 1969. A4). *Show that* $\displaystyle\int_0^1 x^x \, dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}$.

Problem 3 (Putnam 1977. A4). *For $0 < x < 1$, express* $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$ *as a rational function of x.*

Problem 4 (Putnam 2012. A2). *Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.*

Problem 5 (Putnam 2022. A2). *Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?*

Problem 6 (Putnam 2023. A6). *Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon$ the number k was chosen on the kth turn$\}$ matches his goal. For which values of n does Bob have a winning strategy?*

Hints:

1. Does a direct induction argument work?
2. Write $x^x = e^{x \ln x}$ and then use a Taylor series expansion.
3. Express each term as a power series in x, then add them together.
4. If S were a group, then you could multiply $a*c = b*c$ by c^{-1} to get the result. Pretend S is a group: then the element $x = c^{-1}$ would satisfy $(a * c) * x = a$, and then the identity would be $e = c * x$. Now define x and e this way, and work out $a * e$ and $b * e$.
5. Clearly the degree-0 and degree-$2n$ coefficients are nonnegative. Is it possible for all of the rest to be negative? What sign conditions would that impose on the coefficients of the original polynomial?
6. Split cases based on whether n is even or odd. If n is even look for a move-pairing argument, and if n is odd look for an induction argument starting with $n = 3$.