Life is more fun if you play games. – Roald Dahl

Problem 1 (Putnam 1965. B2). *In a round-robin tournament with n players $P_1, \ldots, P_n$, each player plays one game with each of the other players, and no ties can occur. Let w_r and l_r be the number of games won and lost, respectively, by P_r. Show $\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} l_r^2$.*

Problem 2 (Putnam 1995. B5). *A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either (a) one bean from a heap, provided at least two beans are left behind in that heap, or (b) a complete heap of two or three beans. The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.*

Problem 3 (Putnam 2002. A4). *In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?*

Problem 4 (Putnam 2006. A2). *Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)*

Problem 5 (Putnam 2014. B5). *Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are: (1) A player cannot choose an element that has been chosen by either player on any previous turn, (2) A player can only choose an element that commutes with all previously chosen elements, (3) A player who cannot choose an element on his/her turn loses the game. Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)*

Problem 6 (Putnam 2020. B2). *Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?*

Hints:

1. Subtract one side from the other and then factor.

2. Call a heap an even number of moves away from empty "even" and an odd number of moves away from a win "odd". (Thus, heaps of size 0 are even, while size 2,3 are odd, 4 is even, etc.). Every move switches parity, except a move that takes a 3 to a 2.

3. Player 0 wins by making a row or column of 0s, or by making a 2x2 "square" of 0s. Player 1 wins otherwise. You can assume player 1 starts in the upper left: have 0 take the center, and look at the possible responses by 1.

4. If not, then for all $n \geq N$, Alice wins the game with n stones. But Alice's first move must be to a game where the first player loses, which is to say, Alice must leave at most N stones after her first move. What does this say about primes?

5. If P chooses the matrix M on his first move, then all of the subsequent matrices must commute with M. For P to be able to win, there must be an odd number of such matrices – show in fact that P can win if and only if this number is odd. (Hint for that: Lagrange's theorem and Cauchy's theorem.)

6. Try the game for small k first, such as $k = 2$. Look for a parity argument when $k = 2$, to pair up the peg holes in such a way that Bob can always respond to Alice's move. Then generalize.