The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. – John von Neumann

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Problem 1 (Putnam 2014. B2). *Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?*

Problem 2 (Putnam 2021. A2). *For every positive real number x, let $g(x) = \lim_{r\to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}$. Find $\lim_{x\to\infty} \frac{g(x)}{x}$.*

Problem 3 (Putnam 2021. B1). *Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?*

Problem 4 (Putnam 2020. A3). *Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether $\sum_{n=1}^{\infty} a_n^2$ converges.*

Problem 5 (Putnam 2018. A3). *Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.*

Problem 6 (Putnam 2016. B6). *Evaluate* $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Hints:

1. Since $1/x$ is decreasing, the beginning of the interval $[1, 3]$ "counts more" than the end of the interval. In order to have $\int_1^3 f(x)\,dx = 0$, where should f be positive and where should it be negative?

2. Note $\ln g(x) = \lim_{r\to 0} \dfrac{\ln[(x+1)^{r+1} - x^{r+1}]}{r}$, and now try L'Hopital's rule.

3. Suppose the dropped square makes an angle θ with the positive x-axis, for $0 \le \theta \le \pi/2$. What is the area of the region where the center can be dropped so that the square doesn't contain any vertex?

4. The approximation $\sin x \approx x$ clearly isn't good enough here. How about $\sin x \approx x - x^3/6$? (If you plug in $x = 1/\sqrt{n}$, how big is $\sin x$? Is it bigger than $1/\sqrt{n+1}$?)

5. Use the triple-angle formula to turn it into a polynomial min-max question.

6. Notice that $\frac{1}{k2^n+1} = \int_0^1 x^{k2^n}\,dx$. Start interchanging sums and integrals, and simplify the results.