Counting Number Field Extensions of Given Degree, Bounded Discriminant, and Specified Galois Closure

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August 7, 2014

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- $\mathrm{Nm}_{K/\mathbb{Q}}$ be the absolute norm on ideals,
- The Galois closure of L/K be $\hat{L}/K$.

Counting Functions

Define $N_{K,n}(X; G)$ to be the number of number fields L (up to K-isomorphism) such that

- $[L : K] = n$,
- The discriminant norm $\mathrm{Nm}_{K/\mathbb{Q}}(D_{L/K})$ is less than X, and
- The Galois group $\mathrm{Gal}(\hat{L}/K)$ is permutation-isomorphic to G.

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Also define $N_{K,n}(X)$ to be the number of extensions satisfying the first two conditions above (i.e., with no condition on the Galois group).

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For a given K and n, how fast does $N_{K,n}(X)$ grow as X grows?

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This result is known to hold for $n \leq 3$ for general base fields, and for $n \leq 5$ over $\mathbb{Q}$: these results are due to Davenport-Heilbronn, Datskovsky-Wright, Kable-Yukie, and Bhargava.

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Theorem 4 (Ellenberg, Venkatesh)

For all $n > 2$ and all base fields K,

$$N_{K,n}(X) \ll (X\, D_{K/\mathbb{Q}}^{n}\, A_n^{[K:\mathbb{Q}]})^{\exp(C\sqrt{\log n})},$$

where A_n is a constant depending only on n and C is an absolute constant.

More Conjectures

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Conjecture 6 (Malle, weak form)

For any $\epsilon > 0$,

$$N_{K,n}(X; G) \ll X^{a(G)+\epsilon}$$

where $0 < a(G) \leq 1$ is a computable constant depending on G contained in $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

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This conjecture (or a stronger version) is known in a number of cases: for example, if $n \leq 4$, or if G is a nilpotent group.

Counting by Discriminant

Theorem 7 (D.)

Let $n \geq 2$, let K be any number field, and let G be a proper transitive subgroup of S_n. Also, let t be such that if G' is the intersection of a point stabilizer in S_n with G, then any subgroup of G properly containing G' has index at least t. Then for any $\epsilon > 0$,

$$N_{K,n}(X;G) \ll X^{\frac{1}{2(n-t)}\left[\sum_{i=1}^{n-1} \deg(f_{i+1}) - \frac{1}{[K:\mathbb{Q}]}\right]+\epsilon},$$

where the f_i for $1 \leq i \leq n$ are a set of "primary invariants" for G, whose degrees (in particular) satisfy $\deg(f_i) \leq i$.

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- There exist elements $f_1, \cdots, f_n \in R$ such that R is a finitely-generated module over $A := \mathbb{C}[f_1, \cdots, f_n]$. These polynomials are called "primary invariants" of ρ.
- Moreover, there exist polynomials $g_1, g_2, \cdots, g_k \in R$ such that $R = A \cdot g_1 + \cdots + A \cdot g_k$; these polynomials are called "secondary invariants" of ρ.

Primary Invariants, II

Example

Let $G = S_n$ and ρ be the regular representation of G (which simply acts on $\mathbb{C}[x_1, \cdots, x_n]$ by index permutation). Then the elementary symmetric polynomials are a set of primary invariants for G.

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In fact, the elementary symmetric polynomials are a set of primary invariants for any permutation representation... but not necessarily of minimal degree!

Primary Invariants, III

Example

Let $G = \langle(1\,2\,3\,4\,5\,6\,7),\ (1\,2)(3\,6)\rangle \cong PSL_2(\mathbb{F}_7)$*, with* ρ *the natural permutation representation. The following polynomials are a set of primary invariants for* ρ*:*

$$\begin{aligned}
f_1 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
f_2 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \\
f_3 &= x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 \\
f_4 &= x_1x_2x_3 + [\mathit{26\ more\ terms}] + x_5x_6x_7 \\
f_5 &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 \\
f_6 &= x_1^2x_2x_3 + [\mathit{82\ more\ terms}] + x_5x_6x_7^2 \\
f_7 &= x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7 + x_7^7
\end{aligned}$$

Primary Invariants, IV: A New Hope

By the previous slide, we know that if G is the simple group of order 168 and ρ is its permutation embedding in S_7, then ρ has a set of primary invariants of degrees 1, 2, 3, 3, 4, 4, and 7. For this group, one can also check that the t-parameter is equal to 1. Therefore, Theorem 7 yields the following:

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Corollary 8

If G is the simple group of order 168, embedded in S_7, then $N_{\mathbb{Q},7}(X; G) \ll X^{11/6+\epsilon}$.

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For comparison, Schmidt's bound gives the weaker upper bound of $\ll X^{9/4}$, whereas Malle's conjecture posits that the actual count is $\ll X^{1/2+\epsilon}$.

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- Use the invariant theory of G to construct a finite scheme map to affine space.
- Count integral scheme points whose images lie in an appropriate box, to obtain an upper bound on the number of possible α and hence the number of possible extensions L/K.

Transitive Subgroups of S_7

Here are the results of Theorem 7 for transitive subgroups of S_7:

#	Ord	Isom to	Invariant Degs.	Result	Malle	Schmidt
7T1	7	C_7	1,2,2,2,3,4,7	$X^{19/12}$	$X^{1/6}$	$X^{9/4}$
7T2	14	D_7	1,2,2,2,3,4,7	$X^{19/12}$	$X^{1/3}$	$X^{9/4}$
7T3	21	F_{21}	1,2,3,3,3,4,7	$X^{7/4}$	$X^{1/4}$	$X^{9/4}$
7T4	42	F_{42}	1,2,3,3,4,6,7	X^{2}	$X^{1/3}$	$X^{9/4}$
7T5	168	$PSL_2(\mathbb{F}_7)$	1,2,3,3,4,4,7	$X^{11/6}$	$X^{1/2}$	$X^{9/4}$
7T6	2520	A_7	1,2,3,4,5,6,7	$X^{13/6}$	$X^{1/2}$	$X^{9/4}$

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7T4	42	F_{42}	1,2,3,3,4,6,7	X^2	$X^{1/3}$	$X^{9/4}$
7T5	168	$PSL_2(\mathbb{F}_7)$	1,2,3,3,4,4,7	$X^{11/6}$	$X^{1/2}$	$X^{9/4}$
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For horizontal brevity, the results appear without the $+\epsilon$ term in the exponent, and are also stated for the base field $K = \mathbb{Q}$. A superior bound is available for the cyclic and dihedral groups (the former is abelian, while dihedral extensions can be bounded using class field theory).

Towards a Generalization

We now turn to generalizing Theorem 7 into a setting with arbitrary representations ρ, rather than merely permutation representations: so let ρ be a faithful d-dimensional representation of G that is defined over $\mathcal{O}_K$, and $\hat{L}/K$ be a Galois extension.

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Many of the ingredients (invariant theory, point-counting) carry through to the general setting... but it turns out that we will need to define a new counting function first.

Compatible Actions and the Tuning Submodule

There are two natural actions of G on the space

$$\mathcal{O}_{\hat{L}} \otimes_{\mathcal{O}_K} \mathcal{O}_K^{\oplus d} \cong \mathcal{O}_{\hat{L}}^{\oplus d}$$

namely, the action δ arising from the Galois action of G on $\mathcal{O}_{\hat{L}}$, and the action τ arising from the representation ρ.

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namely, the action δ arising from the Galois action of G on $\mathcal{O}_{\hat{L}}$, and the action τ arising from the representation ρ.

Definition

We define the tuning submodule Ξ_ρ to be the subset of elements of the space $\mathcal{O}_{\hat{L}}^{\oplus d}$ on which the two actions δ and τ coincide; namely,

$$\Xi_\rho = \left\{ x \in \mathcal{O}_{\hat{L}}^{\oplus d} : \forall g \in G,\ \delta(g)(x) = \tau(g)(x) \right\}.$$

The tuning submodule Ξ_ρ is a torsion-free $\mathcal{O}_K$-module of rank d.

The ρ-Discriminant

In our new setting, we will use the tuning submodule to obtain a lattice. To do this, we tensor with $\mathbb{R}$ to embed Ξ_ρ inside an ambient real space. The covolume of the resulting lattice yields a natural analogue of the discriminant:

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Definition

If Ξ_ρ is the tuning submodule attached to $(\rho, \hat{L}, K)$, and

$$\psi : \Xi_\rho \to \Xi_\rho \otimes_{\mathbb{Z}} \mathbb{R}$$

is the natural embedding, we define the ρ-discriminant $D_{L/K}^{(\rho)}$ to be

$$D_{\hat{L}/K}^{(\rho)} = \mathrm{covol}(\psi(\Xi_\rho))^2.$$

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Let $K = \mathbb{Q}$, $\hat{L} = \mathbb{Q}(\sqrt{D})$, and ρ be the nontrivial 1-dimensional representation of $G = \mathbb{Z}/2\mathbb{Z}$. For g the nonidentity element of G and $x = a + b\sqrt{D} \in \mathcal{O}_{\hat{L}}$, we have

$$\delta(g)(x) = a - b\sqrt{D}$$

and

$$\tau(g)(x) = -x = -a - b\sqrt{D}.$$

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Therefore,

$$\Xi_\rho = \mathbb{Z}\sqrt{D},$$

and

$$D^{(\rho)}_{\hat{L}/\mathbb{Q}} = D.$$

A New Counting Function

Define $N_{K,n}(X;\rho)$ to be the number of number fields $\hat{L}$ (up to K-isomorphism) such that

- $[L:K]=n$,
- The Galois group $\mathrm{Gal}(\hat{L}/K)=G$ and $\rho: G\to GL_d(\mathcal{O}_K)$ is a faithful representation of G, and
- The ρ-discriminant $D^{(\rho)}_{\hat{L}/K}$ is less than X.

Counting by ρ-Discriminant

We can now state our generalization of Theorem 7:

Theorem 9 (D.)

Let K be any number field, G be a finite group of order n, and $\rho : G \to GL_d(\mathcal{O}_K)$ be a faithful d-dimensional representation of G on $\mathcal{O}_K$. Also define $t(\rho)$ to be the smallest positive integer such that for any nontrivial subgroup H of G, $(\mathcal{O}_K^d)^{\rho(H)}$ has rank $\leq t(\rho)$ as an $\mathcal{O}_K$-module. Then

$$N_{K,n}(X;\rho) \ll X^{\frac{1}{2(d-t(\rho))}\left[\sum_{i=1}^{d} \deg(f_i)\right]},$$

where the f_i for $1 \leq i \leq d$ are a set of primary invariants for ρ. Furthermore, if ρ has a nontrivial secondary invariant, then we can replace the upper bound by $X^{\frac{1}{2(d-t(\rho))}\left[\sum_{i=1}^{d} \deg(f_i) - \frac{\deg(f_1)}{2[K:\mathbb{Q}]}\right]+\epsilon}$.

A Simple Example

Let G be the simple group of order 168 (which has irreducible representations of degrees 1, 3, 3, 6, 7, and 8) and let ρ be one of the 3-dimensional representations.

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A (rather involved) computation will show that ρ has a set of primary invariants of degrees 4, 6, and 14. Theorem 9 then implies

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Corollary 10

For G, ρ, and θ as above,

$$N_{\mathbb{Q}(\theta),168}(X;\rho) \ll X^{47/4+\epsilon}$$

.

Future Directions

Some avenues for future work:

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- Strengthen point-counting techniques.
- Establish lower bounds on $N_{K,n}(X;\rho)$.
- Adapt results to other types of extensions (e.g., of function fields).
- Relate ρ-discriminant to other invariants (Artin conductors, the classical discriminant, etc.).

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- You, for attending this talk!