A Tour of Classical and Modern Cryptography

Evan P. Dummit

University of Rochester

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What is Cryptography?

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- ways to send secure messages across an insecure network, or
- ways to send messages to your friends without your enemies reading them, or
- ways to protect sensitive data from attackers.

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Cryptography is generally a blend of three things:

- Mathematics (number theory, linear algebra, combinatorics)
- Computer science and computer engineering
- Cleverness

Goals of Cryptography, I

Cryptography uses standard names:

- Alice has "plaintext" that she wants to encrypt to make "ciphertext".
- Bob receives encrypted ciphertexts from Alice that he wants to decrypt (he may also send messages back).
- Eve is an eavesdropper: she spies on Alice and Bob.

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Important Goal of Cryptography

Alice and Bob want to communicate without Eve being able to decode their messages.

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Eve might also try to alter messages, or impersonate Alice or Bob.

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Bob wants to verify all the messages he receives are really from Alice, and not Eve pretending to be Alice.

Caesar Shift, I

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- Choose a number from 1 to 25.
- To encode, shift plaintext alphabetically forward by that many letters (wrap from Z to A).
- To decode, shift back by the same amount.

Example

Shift by 3 letters: **yellowjackets** $\rightarrow$

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Want to avoid both of these kinds of attacks.

Ultimate issue: Caesar shift is too predictable.

Substitution, I

Reinterpretation of Caesar shift: it replaces each letter of plaintext with another one in some fixed manner. Can generalize with a general "letter substitution":

- To each plaintext letter **a**-**z**, assign a ciphertext letter (using each letter exactly once).
- For example: $\mathbf{a} \mapsto \mathbf{M}$, $\mathbf{b} \mapsto \mathbf{D}$, ... , $\mathbf{z} \mapsto \mathbf{W}$.
- To decode, simply make the replacements in reverse.

Substitution, II

Can summarize substitution by listing "encoding alphabet":

```
abcdefghijklmnopqrstuvwxyz
TFOANYRLEMZSGKPXQCDIHBUJVW
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Example

With the encoding alphabet above,

```
substitution is slightly better than caesar shifting
                           ↓
DHFDIEIHIEPK ED DSERLISV FNIINC ILTK OTNDTC DLEYIEKR
```

Substitution *is* slightly better than Caesar shifting

A bit less obvious how to crack substitution ciphers.

- If given a sample of plaintext + ciphertext, fairly easy to decode remainder (get most of the encoding alphabet).
- But what if just ciphertext is given?
- Brute force has too many possibilities: $26! \approx 4.03 \cdot 10^{26}$.

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Key idea is "frequency analysis". For English:

Letter	e	t	a	o	i	n	s
Frequency	12.7%	9.1%	8.2%	7.5%	7.0%	6.7%	6.3%
Letter	h	r	d	l	c	u	m
Frequency	6.1%	6.0%	4.3%	4.0%	2.8%	2.8%	2.4%

Substitution, IV: A New Hope

Can break substitution ciphers by counting letter frequencies.

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Get extra mileage using frequencies of 2-letter combinations ("digrams") and 3-letter combinations ("trigrams"):

- Digrams: **th**, **he**, **in**, **an**, **er**, **re**, **ed**, **on**, **es**, **ea**, **ti**, **st**, **en**, **at**.
- Trigrams: **the**, **ing**, **and**, **ere**, **her**

Substitution, V

Let's do an example:

KB TI BQ NBK KB TI KRZK PF KRI XAIFKPBN DRIKRIQ KPF NBTHIQ PN KRI YPNC KB FAWWIQ KRI FHPNOF ZNC ZQQBDF BW BAKQZOIBAF WBQKANI BQ KB KZGI ZQYF ZOZPNFK Z FIZ BW KQBATHIF ZNC TJ BEEBFPNO INC KRIY

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```

Quick count: 19 K, 18 B, 17 I, 13 F, ...
Also, 6 each of KR, RI, ..., and 5 KRI.

Substitution, V

Try KRI = the:

```
tB Te BQ NBt tB Te thZt PF the XAeFtPBN DhetheQ tPF
NBTHeQ PN the YPNC tB FAWWeQ the FHPNOF ZNC ZQQBDF BW
BAtQZOeBAF WBQtANe BQ tB tZGe ZQYF ZOZPNFt Z FeZ BW
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Now make educated guesses: how about B = o and Z = a?

Substitution, V

Let's take a look now:

```
to Te oQ Not to Te that PF the XAeFtPoN DhetheQ tPF
NoTHeQ PN the YPNC to FAWWeQ the FHPNOF aNC aQQoDF oW
oAtQaOeoAF WoQtANe oQ to taGe aQYF aOaPNFt a Fea oW
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Now let's try $D = w$, $Q = r$, $F = s$, $P = i$:

Substitution, V

Let's take a look again:

```
to Te or Not to Te that is the XAestioN whether tis
NoTHer iN the YiNC to sAWWer the sHiNOs aNC arrows oW
oAtraOeoAs WortANe or to taGe arYs aOaiNst a sea oW
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```

Clearly, T = b, N = n, X = q, A = u, H = l, Y = m, C = d, …

Substitution, Victory

It's starting to come together now:

`to be or not to be that is the question whether tis nobler in the mind to suffer the slings and arrows of outrageous fortune or to take arms against a sea of troubles and by opposing end them`

Polyalphabetic Ciphers

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Example

```
  plaintext
+ keyletter
-----------
= zpytrmxbk
```

($p = 15$, $k = 10$ *so* $p + k = 25 = z$, *etc.)*

The Vigenère Cipher

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Some people[citation needed] thought this cipher unbreakable.

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Some people[citation needed] thought this cipher unbreakable. (It's not!) Main weakness: the finite keyword makes keytext too predictable.

Running-Key Ciphers

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- One answer: use a book or other common text as the key.

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Works fairly well... but long words in a predictable keytext can be guessed. Still not secure!

One-Time Pads

How to fix predictability of running-key cipher? Use random letters!

Example

```
  samplemessage
+ hqnyjiefsehpb
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= zqznumqjkwhvf
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This turns out to be provably unbreakable... in theory. But some issues in practice:

- Hard for computers and humans to generate random letters.
- Must keep sender + receiver keys secure indefinitely.
- Must use key only once (then destroy it).
- No protection if key is secretly copied.

Other Historical Cryptosystems

Some historically notable cryptosystems used before the 1960s:

- Playfair cipher: invented 1850, used by British in WWI.
- Hill cipher: invented 1929, based on linear algebra. (Little-used.)
- ADFGX/ADFGVX: invented 1918, used by Germans in WWI. Broken by French lieutenant in June 1918.
- Enigma: encoding machines, used by Germans in WWII. Broken by Allied efforts, mostly because of improper use of the system.

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What Alice and Bob need is an "asymmetric" cryptosystem: one where information needed to encode $\neq$ information needed to decode.

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- Basic idea: "one-way function", a function easy to evaluate but hard to invert.
- Many examples come from number theory.
- Simple example: $f(p, q) = pq$, for primes p and q.
- Multiplication is easy: can find $f(331, 443) = 146633$ by hand.
- Factoring is hard: try finding p, q with $f(p, q) = 339281$ by hand.

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- Bob makes encoding mechanism freely available to everyone. (Eve too!)
- Alice uses encoding mechanism to send messages to Bob.
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Please take a moment to reflect on how amazing this idea is, even if you're familiar with it already.

Modular arithmetic, I

Need to do a bit of number theory to go further.

Definition

If m is a positive integer, we say $a \equiv b \pmod{m}$ if m divides $b - a$.

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Example

We have $3 \equiv 21 \pmod{6}$, $11 \equiv -3 \pmod{7}$, and $10^{10} \equiv (-1)^{10} \equiv 1 \pmod{11}$.

Arithmetic behaves normally: we can add, subtract, multiply congruences with the same modulus.

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- Very dumb way: Evaluate 3^{2056}, divide by 22, find remainder.
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- Dumb way: Evaluate 3^{2056} but reduce mod 22 each time.
- Smarter way: Evaluate $3^1, 3^2, 3^4, 3^8, \ldots, 3^{2048} \pmod{22}$ by squaring each time. Then $3^{2056} = 3^{2048} \cdot 3^8$.

Modular arithmetic, III

Recall that two integers are "relatively prime" if they have no common divisors other than ± 1.

Definition

If m is a positive integer, then the Euler φ-function $\varphi(m)$ is defined to be the number of integers k, $1 \leq k \leq m$ relatively prime to m.

Example: $\varphi(10) = 4$ (only 1, 3, 7, 9 are relatively prime to 10).

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Example: $\varphi(10) = 4$ (only 1, 3, 7, 9 are relatively prime to 10).

Fact

If $m = pq$ is a product of two distinct primes, then $\varphi(m) = (p-1)(q-1)$.

(There is a more general formula, but we don't need it.)

Modular arithmetic, IV

Now we can state the key result about powers we need:

Theorem (Euler's Theorem)

If m is a positive integer and a is relatively prime to m, then $a^{\varphi(m)} \equiv 1 \pmod{m}$.

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Example

Euler's Theorem says $7^4 \equiv 1 \pmod{10}$ and $2^{24} \equiv 1 \pmod{35}$.

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- Bob chooses two large primes p and q and computes $N = pq$.
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For Alice to send Bob an encrypted message:

- Alice breaks message into a sequence of nonnegative integers each less than N and encrypts each separately.
- Alice encrypts m as $c \equiv m^e \pmod{N}$ and sends c to Bob.

The RSA Encryption System, II

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- Bob first finds a value d such that $de \equiv 1 \pmod{\varphi(N)}$.
- He can do this very quickly using the Euclidean algorithm.
- To decrypt a ciphertext c, Bob computes $m \equiv c^d \pmod{N}$.

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Here is an example:

- Bob picks $p = 5$ and $q = 11$ and computes $N = 55$.
- Bob also computes $\varphi(N) = 40$ and picks $e = 3$.
- Alice wants to send $m = 7$ to Bob.
- She computes $c = m^e \equiv 13 \pmod{N}$, and sends it.
- Bob then finds $d = 27$ has $de \equiv 1 \pmod{\varphi(N)}$.
- To decrypt, Bob computes $c^d \equiv 7 \pmod{N}$.

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- So $c^d \equiv m^{de} \pmod{N}$.
- Also remember $de \equiv 1 \pmod{\varphi(N)}$, so $de = 1 + k\varphi(N)$ for some k.
- Then $m^{de} \equiv m \cdot (m^{\varphi(N)})^k \equiv m \cdot 1^k \equiv m \pmod{N}$, where we used Euler's theorem to observe that $m^{\varphi(N)} \equiv 1 \pmod{N}$.

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Technically, this explanation only applies when m is relatively prime to N (which is almost always the case), but it can be shown the procedure still works even when m has a common factor with N.

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- Suppose Eve is eavesdropping and records all the data sent, so she has N, e, and c (but not p, q, or d) and wants to find m.
- Eve needs to solve the congruence $m^e = c \pmod{N}$ for m.
- If she could find $\varphi(N)$ then she could compute d the same way Bob did.

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- Eve needs to solve the congruence $m^e \equiv c \pmod{N}$ for m.
- If she could find $\varphi(N)$ then she could compute d the same way Bob did.
- However, if $N = pq$ is a product of two primes, then computing $\varphi(N) = (p-1)(q-1)$ is equivalent to factoring N.
- If Bob picks p and q really large (200 base-10 digits or so), it is extremely hard to factor N.

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- It turns out that the answer is "probably not".
- Specifically, if Eve found a decryption exponent that could decode any ciphertext c, then she could use it to factor N. (Details are a bit too technical to give here.)

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- It turns out that the answer is "probably not".
- Specifically, if Eve found a decryption exponent that could decode any ciphertext c, then she could use it to factor N. (Details are a bit too technical to give here.)
- Of course, this isn't quite what we asked: Eve just wants to decode one message, not all messages.
- But this suggests breaking RSA is roughly the same difficulty as factoring large numbers.

RSA is not known to be equivalent to factoring. (Some other public-key cryptosystems are.)

Analysis of RSA, III

RSA seems secure, at least if factoring large numbers is slow. But why isn't it slow for Alice and Bob too?

- Bob first needs to find two large primes p and q.
- Perhaps counterintuitively, it is much easier to check whether a number is prime than it is to factor it.
- Primes are also easy to find: a random 200-digit integer not divisible by 2, 3, 5, 7 has about a 1% chance of being prime.
- So Bob can generate p and q quickly even if he wants them to have hundreds of digits.

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- So Bob can generate p and q quickly even if he wants them to have hundreds of digits.
- Likewise, Bob can find the decryption exponent d very fast.
- Finally, Alice and Bob can both do encryption and decryption quickly using successive squaring.

Proving and Verifying

Suppose Alice sends Bob a message using RSA, and she wants verification that Bob actually received it. How can this be done?

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Suppose Alice sends Bob a message using RSA, and she wants verification that Bob actually received it. How can this be done?

- Idea 1: Bob sends Alice a response saying "I received your message".
- Issue: Alice has no way to know Bob really sent that message: maybe it was actually Eve.
- Idea 2: Bob sends Alice a response saying "I received your message" and then includes part of Alice's message.
- Issue: Eve might have been impersonating Alice. Now Bob has revealed some of Alice's message to Eve.

Ultimately, how can Bob prove to Alice that he got her message without revealing anything about it? What we want is a "zero-knowledge proof".

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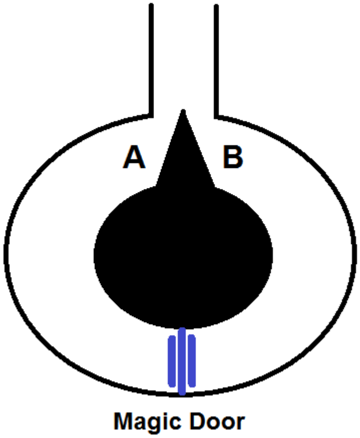

- Imagine a cave with one entrance.
- Entrance splits off to two paths: A and B.
- Paths lead to opposite sides of magic door.

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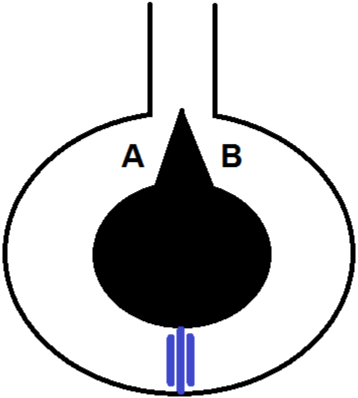
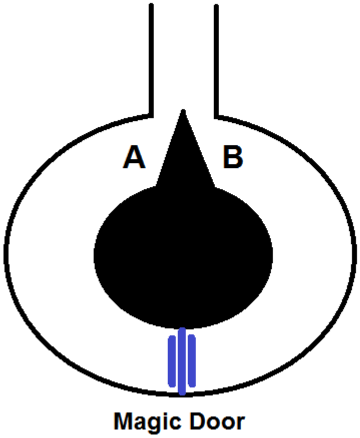


- Imagine a cave with one entrance.
- Entrance splits off to two paths: A and B.
- Paths lead to opposite sides of magic door.
- Peggy claims she knows magic words to open door.
- Victor says “prove it”.
- But Peggy doesn’t want to reveal magic words.

A Zero-Knowledge System, II

How can Peggy prove she knows the magic words to Victor?



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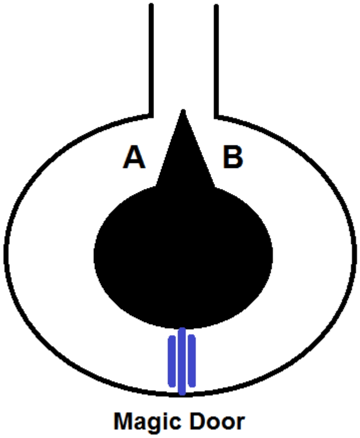

- Peggy walks into cave along A or B. (Victor can't see which.)
- Victor then calls out "A" or "B".
- Peggy walks back along path Victor chose.
- Victor and Peggy repeat procedure 20 times (or until Victor believes Peggy knows secret).

A Zero-Knowledge System, III

Why does this procedure work?

- If Peggy really knows magic words, she passes every time (she just goes through the door if she's on wrong side).
- If Peggy doesn't know magic words, there is 50% chance she is on wrong side and will fail test.

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- But repeating test 20 times means there is only a $1/2^{20} \approx 10^{-6}$ probability Peggy doesn't really know magic words but still passed.

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- For Victor, one single test doesn't give much confidence.
- But repeating test 20 times means there is only a $1/2^{20} \approx 10^{-6}$ probability Peggy doesn't really know magic words but still passed.
- What about if Eve is watching them?
- Maybe Peggy and Victor conspired together to fake the tests (they just decide ahead of time what Victor will say).
- Eve shouldn't be convinced Peggy knows magic words.

Implementation of Zero-Knowledge System, I

How can we implement this idea? Can do it a bit like RSA:

- Peggy chooses two large primes p and q and a secret s.
- Peggy publishes $N = pq$ and $s^2 \pmod{N}$.
- Victor wants Peggy to prove she knows s.

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Here is the test procedure:

- Peggy chooses a random number r modulo N and sends Victor the value $r^2 \pmod{N}$.
- Victor then asks either for r or for rs modulo N.
- Peggy sends Victor the quantity he requested.
- If Victor asked for r, he checks whether the value squares to r^2 Peggy sent earlier.
- If Victor asked for rs, he checks whether the value squares to r^2s^2 (which he can compute using r^2 and s^2).

Implementation of Zero-Knowledge System, II

For example, suppose Peggy chooses $N = 264389$, $s = 110296$.
She publishes $s^2 \equiv 140948 \pmod{N}$.
Victor asks Peggy to prove she knows s.

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- Peggy generates $r = 83924$ and sends Victor $r^2 \equiv 179205$.
- Victor says "send me r".
- Peggy sends Victor $r = 83924$, and he verifies $83924^2 \equiv 179205$. Peggy passes this round.

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Round 2:

- Peggy generates $r = 101041$ and sends Victor $r^2 \equiv 166835$.
- Victor says "send me rs".
- Peggy sends Victor $rs \equiv 157397$, and Victor checks $157397^2 \equiv 166835 \cdot 140948$. Peggy passes this round too.

Implementation of Zero-Knowledge System, III

Now suppose Eve tries to impersonate Peggy. Victor asks Eve to prove she knows s.

Implementation of Zero-Knowledge System, III

Now suppose Eve tries to impersonate Peggy. Victor asks Eve to prove she knows s. Round 1:

- Eve knows $N = 264389$ and $s^2 \equiv 140948 \pmod{N}$ but not s itself.
- Eve guesses Victor will ask for r. Eve generates $r = 197866$ and sends Victor $r^2 \equiv 230836$.

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- Victor says "send me r".
- Eve sends Victor $r = 197866$, and he verifies $197866^2 \equiv 230836$.
- Eve passes this round.

Implementation of Zero-Knowledge System, IV

Round 2:

- Eve guesses Victor will ask for r again.
- Eve generates $r = 153819$ and sends Victor $r^2 \equiv 113151$.
- Victor says "send me rs".

Implementation of Zero-Knowledge System, IV

Round 2:

- Eve guesses Victor will ask for r again.
- Eve generates $r = 153819$ and sends Victor $r^2 \equiv 113151$.
- Victor says "send me rs".
- Eve is stuck! She can't send rs because she only knows r and s^2, but not s.
- Whatever Eve sends, Victor will then know Eve didn't really have the value of s.

Analysis of Zero-Knowledge System

Why does this system work? Same as cave example:

- Peggy knows s, so she can always answer Victor's challenges.
- Eve doesn't know s. She only has a 50% chance to answer correctly.
- But why can't Eve find s from s^2 and N?

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- But why can't Eve find s from s^2 and N?
- Answer: being able to compute square roots mod N is equivalent to factoring $N = pq$. Factoring is hard, so Eve probably can't do this.
- Finally, Eve doesn't gain any information from spying on Peggy and Victor, since r and rs are never both sent during a single round.

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Now that you know a few of the ideas, how can we use them? A few examples:

- Sending secure communications across insecure channels.
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- Designing secure digital voting systems.
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There are lots more, but I think I'll stop here.