Points, Lines, and Dimensions: A Tour of The Kakeya Problem

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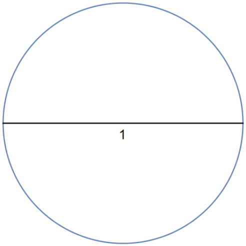
The Kakeya Needle Problem, I

Definition (S. Kakeya; 1917)

A **Kakeya needle set** *is a subset of the plane inside which it is possible to rotate a needle of length 1 completely around.*

An example: a circle of diameter 1 (area $\pi/4$):

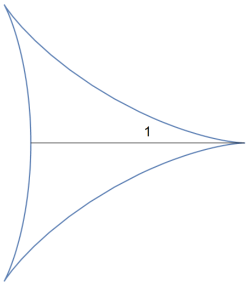
Circle as a Kakeya Set



The Kakeya Needle Problem, II

Another example: a deltoid (area $\pi/8$):

Deltoid as a Kakeya Set



The Kakeya Needle Problem, III

Question

What is the minimum area of a Kakeya needle set?

It was originally believed that the deltoid example (of area $\pi/8$) was the smallest possible Kakeya set. But....

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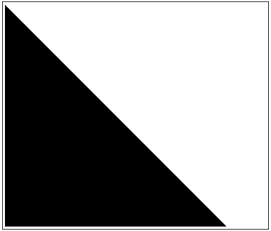
Theorem (A. Besicovitch; 1919)

There exists a Kakeya needle set in the plane having arbitrarily small area.

The Kakeya Needle Problem, IV

Basic idea for constructing a Kakeya set of small area:

- Start with a simple Kakeya set.
- Slice up the set into pieces.
- Slide the the pieces together so that they overlap a lot.
- Repeat steps 2-3 until the set is arbitrarily small.



The Kakeya Needle Problem, V

What about higher dimensions? The key idea is that we can orient the needle in any direction:

Definition

For $n \geq 2$, a **Kakeya set** *is a set in $\mathbb{R}^n$ inside which it is possible to rotate a needle of length 1 to point in any direction.*

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We can get Kakeya sets having arbitrarily small volume in $\mathbb{R}^n$ simply by taking a Cartesian product of $[0,1]^{n-2} \times K$, where K is a Kakeya set in the plane.

Besicovitch and Kakeya Sets

The needle moves continuously, so we can't ever expect to get a Kakeya set of area zero (though this isn't quite so trivial to prove!). So, let's modify the definition slightly.

Definition

A **Besicovitch set** *is a set of points in Euclidean space which contains a unit line segment in every direction.*

Any Kakeya set is certainly a Besicovitch set, but....

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Theorem

There exists a Besicovitch set in the plane of area 0.

To justify this requires a more careful definition of "area", but the idea is simply to take a limit in the construction described earlier.

Dimension, I

Besicovitch sets can be very small in area. But there are other notions of size!

Definition

The **Minkowski dimension** *of a set K is defined to be*

$$\dim(K) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

where $N(\epsilon)$ is the number of boxes of side ϵ needed to cover K.

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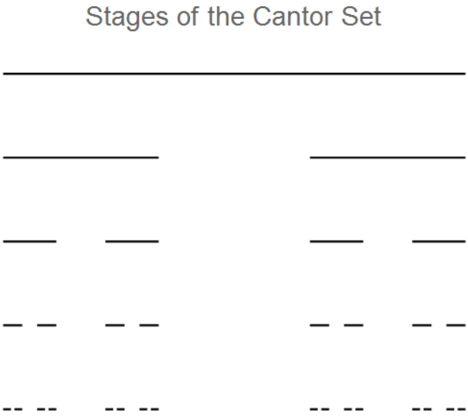
Motivation: how many ϵ-scale boxes do we need to cover the set? For a line, ϵ^{-1}; for a square, ϵ^{-2}, for a cube, ϵ^{-3}, and so forth.

Dimension, II

We can compute dimensions of other sets, like the famous "Cantor set": starting with the interval $[0, 1]$, remove the middle third, and then repeatedly remove the middle third of each remaining piece:



Dimension, III

We can find the Minkowski dimension of the Cantor set by counting how many "boxes" (in this case, segments) of various lengths we need to cover it:

ϵ	$N(\epsilon)$	$\log(N(\epsilon))/\log(1/\epsilon)$
$1/10$	8	0.903
$1/100$	32	0.753
$1/1000$	128	0.702
$1/10000$	512	0.677
$1/10^6$	2^{13}	0.652
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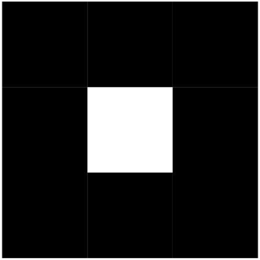
If the length is $\epsilon = 1/3^n$, then we require $N(\epsilon) = 2^n$ segments. The Minkowski dimension is $\log(2^n)/\log(3^n) = \log_3 2 \approx 0.631$.

Dimension, IV

The Cantor set has dimension between 0 and 1: it has "fractional dimension" = a "fractal". We can perform similar calculations for other famous fractals: here is one called the "Sierpinski carpet":

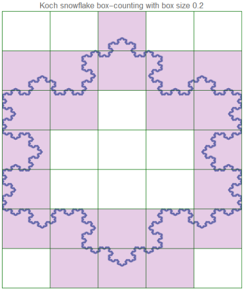
Sierpinski Carpet, Stage 1

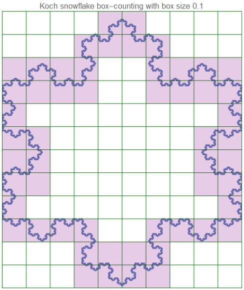


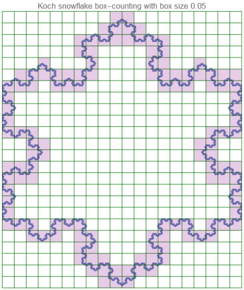
By using boxes of side length $1/3^n$, try to show that this fractal has Minkowski dimension $\log_3 8 \approx 1.893$.

Dimension, V

Here is another fractal, the "Koch snowflake":







Dimension, VI

Explicit counts are as follows:

ϵ	$N(\epsilon)$	$\log(N(\epsilon))/\log(1/\epsilon)$
$1/5$	21	1.89
$1/10$	54	1.732
$1/20$	129	1.622
$1/100$	922	1.482
$1/1000$	10765	1.344
$1/10000$	154046	1.297

The actual dimension of the Koch snowflake turns out to be $\log_3(4) \approx 1.261$.

Dimension and Kakeya, I

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Dimension and Kakeya, I

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Theorem (R. Davies; 1971)

Any Besicovitch or Kakeya set in $\mathbb{R}^2$ has Minkowski dimension 2.

What about in higher dimensions?

Conjecture (Kakeya Conjecture)

Any Kakeya set in $\mathbb{R}^n$ is of Minkowski dimension n.

Unfortunately, we only have lower bounds when $n > 2$.

Dimension and Kakeya, II

At this point, you might wonder: who is interested in this problem?

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Pictured: Terence Tao, IMO gold medalist (age 13), Princeton PhD (age 21), UCLA Professor (age 24), Fields Medalist (age 31), coauthor of over 300 papers and 17 books.

Dimension and Kakeya, III

Here is what we know about Besicovitch sets in higher dimensions:

Theorem (T. Wolff; 1995)

Any Besicovitch or Kakeya set in $\mathbb{R}^n$ has Minkowski dimension at least $(n+2)/2$.

Dimension and Kakeya, III

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Theorem (T. Wolff; 1995)

Any Besicovitch or Kakeya set in $\mathbb{R}^n$ *has Minkowski dimension at least* $(n+2)/2$.

This was improved for $n > 4$:

Theorem (N.H. Katz, T. Tao; 1995)

Any Besicovitch or Kakeya set in $\mathbb{R}^n$ *has Minkowski dimension at least* $(1/\alpha)n + (1-\alpha)/\alpha \approx 0.596n + 0.403$, *where* $\alpha^3 - 4\alpha + 2 = 0$.

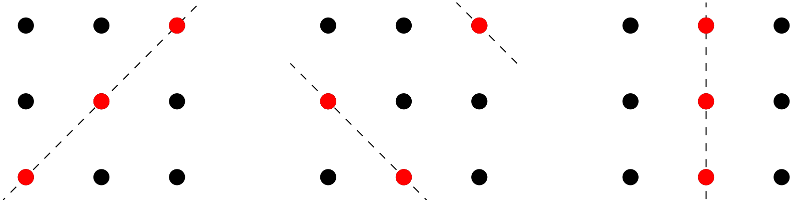
Kakeya Sets Modulo p, I

The proofs of the theorems about dimensions of Besicovitch sets are very hard. So instead of talking about that, let's change the problem!

Specifically, let's try to pose the Kakeya problem "modulo p", by replacing the real numbers $\mathbb{R}$ with the integers modulo p (which we write as $\mathbb{F}_p$), so we are now in the space $\mathbb{F}_p^n$.

Kakeya Sets Modulo p, II

For example, if $n = 2$, then we are simply looking at a $p \times p$ grid of points, where things "wrap around":



Kakeya Sets Modulo p, III

Kakeya sets are defined using points, lines, and distances. So what is a "line modulo p"?

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Definition

The **line** *in* $\mathbb{F}_p^n$ *through the point* $\mathbf{a}$ *with direction vector* $\mathbf{v}$ *consists of the points of the form* $P = \mathbf{a} + t\mathbf{v}$ *for* $t = 0, 1, \ldots, p-1$.

Each line contains p points. In the plane, there are $p+1$ possible directions (corresponding to the possible slopes of a line, including ∞ for vertical lines).

Kakeya Sets Modulo p, IV

What about distances? It turns out that distances are hard to work with modulo p. Fortunately, we can simply discard them!

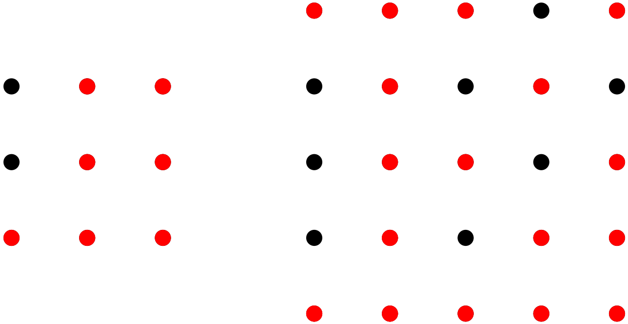
Definition

A **Kakeya set** *is a set of points in $\mathbb{F}_p^n$ that contains a line in every possible direction.*

By "contains a line" we mean "contains the p points on the line". We dispense with the "length 1" part because everything is finite.

Kakeya Sets Modulo p, V

Here are some examples of Kakeya sets, in $\mathbb{F}_3^2$ and $\mathbb{F}_5^2$:



Kakeya Sets Modulo p, VI

So how small can a Kakeya set in $\mathbb{F}_p^n$ be?

Proposition

Any Kakeya set in $\mathbb{F}_p^2$ contains at least $\frac{1}{2}p^2$ points.

Kakeya Sets Modulo p, VI

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Proposition

Any Kakeya set in $\mathbb{F}_p^2$ contains at least $\frac{1}{2}p^2$ points.

Proof: The first line has p points, the second adds at least $p-1$ new points, the third adds at least $p-2$ more, ... , yielding at least $p+(p-1)+\cdots+1 = \dfrac{p(p+1)}{2} > \dfrac{1}{2}p^2$ points in total. $\square$

Reframing: a Kakeya set in $\mathbb{F}_p^2$ contains a positive proportion (namely, at least half) of the points in $\mathbb{F}_p^2$. We can think of this as being like "Minkowski dimension 2".

Kakeya Sets Modulo p, VII

What do we expect for larger n?

Conjecture (Mod-p Kakeya Conjecture, T. Wolff; 1999)

Any Kakeya set in $\mathbb{F}_p^n$ contains at least $c_n p^n$ points, for some constant $c_n > 0$.

This problem seemed as hard as Kakeya in $\mathbb{R}^n$:

Theorem (G. Mockenhaupt, T. Tao; 2004)

Any Kakeya set in $\mathbb{F}_p^n$ contains at least $c_n p^{(4n+3)/7}$ points, for a constant $c_n > 0$.

Their proof is quite intricate and seemed difficult to improve upon.

Kakeya Sets Modulo p, VIII: The Last Jedi

Theorem (Z. Dvir; 2008)

Any Kakeya set in $\mathbb{F}_p^n$ contains at least $\binom{n+p-1}{n} \geq \dfrac{p^n}{n!}$ points.

In other words, a Kakeya set in $\mathbb{F}_p^n$ always has Minkowski dimension n, and contains a positive proportion of the points in $\mathbb{F}_p^n$, even as p grows arbitrarily large. Thus, the mod-p Kakeya conjecture is true.

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It might seem that Dvir's proof must be very complicated, but in fact, we will go through it!

Dvir's Proof, I

Dvir's proof goes as follows:

- Suppose K is a Kakeya set in $\mathbb{F}_p^n$ having $< \binom{n+p-1}{n}$ points.
- Consider the collection of polynomials in n variables $x_1, \ldots, x_n$ of degree at most $p-1$, whose coefficients are considered "modulo p".
- By the famous "stars and bars" counting argument[1], there are $\binom{n+p-1}{n}$ possible monomial terms in such a polynomial.
- Since each monomial term is "independent", there must be some nonzero polynomial P that vanishes at each point in K.

[1] See problem 36 of the 2016 UVM High School Math Exam, or problem 2 from the January 2018 Vermont Math Talent Search.

Dvir's Proof, II

- Let $P = P_0 + P_1 + \cdots + P_{p-1}$ where P_i is homogeneous of degree i.
- For any nonzero $\mathbf{v}$, because P vanishes on a line in the direction $\mathbf{v}$, there exists some $\mathbf{a}$ such that $P(\mathbf{a} + t\mathbf{v}) = 0$ for $t = 0, 1, 2, \ldots, p-1$.
- Then $P(\mathbf{a} + t\mathbf{v})$ is a polynomial of degree at most $p-1$ in the variable t having p distinct roots modulo p, so it must be the zero polynomial.
- By multiplying out, one can verify that the coefficient of t^{p-1} in $P(\mathbf{a} + t\mathbf{v})$ is $P_{p-1}(\mathbf{v})$.
- Therefore, $P_{p-1}(\mathbf{v}) = 0$ for all $\mathbf{v}$ in $\mathbb{F}_p^n$, hence $P_{p-1} = 0$.
- Repeat for the other terms, to conclude that all terms of P are zero. This is a contradiction!

The Polynomial Method

Dvir's proof is a stunning example of the "polynomial method": consider a polynomial vanishing on the set, and then prove something about it. Other applications of the polynomial method:

- Sizes of cap sets (sets avoiding 3-term arithmetic progressions, made famous in the card game "Set").
- Erdős distinct distances problem: given n points in the plane, what is the smallest number of distinct distances between the points in terms of n? (Answer: $\geq cn/\log(n)$ for some $c > 0$.)
- Finite-field Nikodym problem, joints problem, and other variations on point-line configurations.

Sizes of Kakeya Sets Mod p

Distressing caveat: Dvir's proof gives no real information about what Kakeya sets actually look like!

Some improvement in the bound is available, using a slightly more complicated version of the technique:

Theorem (Z. Dvir, S. Kopparty, S. Saraf, M. Sudan; 2009)

For large enough n, a Kakeya set in $\mathbb{F}_p^n$ contains at least $(0.4999)^n p^n$ points.

The 0.4999 in the theorem can be replaced with any number less than 1/2. (It is also believed that this is the best possible constant.)

Open Questions

Kakeya sets have several applications in other mathematical problems (too complicated to describe here).

Here are a few broad questions to think about:

- Can we study Kakeya sets in other settings? (One possibility: modulo m where m is not a prime.)
- Can we find analogies between the results about Kakeya sets in these different settings?
- What if we use shapes other than lines to create Kakeya sets?

End of Talk

Thank you!

(And congratulations again to all of the Math Day winners!)