p-adic Modular Forms, Galois Representations, and Hida Families

Evan, part 2

November 29, 2010

1 History / Background / Goals

- Classical theory dates to early 1800s, connected with elliptic functions.
- Developed in the 1920s by Hecke and others. Classical modular forms live entirely over $\mathbb{C}$.
- Serre, 1970s, developed ad hoc theory of modular forms in a p-adic setting, by p-adically interpolating Eisenstein series.
- Theory of more general p-adic modular forms was subsequently developed "more properly" by Katz and Dwork who did significant work in this area in the mid-1970s.
- Hida expanded more in mid 80s, but still had some limitations. Specifically, Hida constructed families (now called Hida families) of p-adic cuspforms varying with the weight k which were eigenforms for the (p-adic) Hecke operators. In analogy with classical theory, also got Galois representations.
- Mazur, Wiles, and others worked in late 80s and 90s to remove some limitations on Hida's constructions, "eigencurves".
- This talk is following the first half of a short expository paper, "p-adic families of modular forms" by M. Emerton.
- I'll give a crash course in the classical theory, and then basically say everything again in the p-adic setting. In theory our speaker on Thursday will give you more examples in the p-adic setting than I will – my goal is mostly to try to help you see how the results look similar to the classical ones.
- Hopefully by the end of this quick talk you'll have a vague idea of what a system of Hecke eigenvalues is and some idea of how it gives rise to a Galois representation, both in the classical setting and in the p-adic setting.

2 Crash Course in Classical Theory

2.1 Modular Forms

- Holomorphic will always mean holomorphic on the upper half-plane.
- Let $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_1(N) = \left\{\gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}$, $\Gamma_0(N) = \left\{\gamma \in \Gamma : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\}$.
- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, f holomorphic, the weight-k <u>slash operator</u> is $(f\,|_k\,\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma\tau)$, where k, $N \in \mathbb{N}$. Have a natural definition of invariance under this action.
- A <u>modular form</u> (resp. <u>cuspform</u>) of weight k and level N is a holomorphic f invariant under the weight k action of $\Gamma_1(N)$ and which is "holomorphic at ∞" (resp. vanishes at ∞) – i.e., such that $\lim_{y\to\infty}(f\,|_k\,\gamma)$ is finite (resp. 0) for all $\gamma \in SL_2(\mathbb{Z})$.
- Since in particular a modular form of any weight and level is invariant under translation by 1, it has a Fourier decomposition $f(\tau) = \sum_{n=0}^{\infty} c_n(f) \cdot q^n$where $q = e^{2\pi i\tau}$. [Holomorphicity implies that the negative coefficients vanish; a cuspform also has zero constant term.]

- Collection of modular forms of wt. k level N is $\mathcal{M}_k(N)$, and of cuspforms is $\mathcal{S}_k(N)$. They are finite-dimensional vector spaces.
  - Constant functions are the only forms of weight 0. Generally we will assume that $k \geq 1$ since 0 is boring.
  - Eisenstein series $E_{2k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{1}{(m+n\tau)^{2k}}$ for $k > 2$ give simple examples of positive weight.

2.2 Hecke Operators

- Have $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$, and the map sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to d is an isomorphism of the quotient with $(\mathbb{Z}/N\mathbb{Z})^\times$. The wt-$k$ action of $\Gamma_0(N)$ preserves $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ hence we get an action of the quotient on each.
- For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ we write the corresponding automorphism of $\mathcal{M}_k(N)$ as $\langle d \rangle$, sometimes called the "diamond operator".
- If l is a prime not dividing N we define the automorphism $S_l = \langle l \rangle \cdot l^{k-2}$. [This is the l-diamond operator on f, times l^{k-2}.]
- The lth Hecke endomorphismT_l on $\mathcal{M}_k(N)$ is defined as $(T_l\, f)(\tau) = \sum_{n=0}^{\infty} c_{l\cdot n}(f) q^n + \sum_{n=0}^{\infty} l \cdot c_n(S_l f) q^{l\cdot n}$. [Not obvious that this actually preserves $\mathcal{M}_k(N)$, but it does, and $S_k(N)$ too.]
  - Lots of other ways to define the Hecke operator T_l: geometrically it is essentially a sum over all lattices of index l.
  - Algebraically it can be written as a particular double coset operator.
  - Definition extends to composite values of l fairly easily, but won't present for brevity.
- The Hecke algebra $\mathbb{T}_k(N)$ of weight k and level N is the $\mathbb{Z}$-subalgebra of $\mathrm{End}(\mathcal{M}_k(N))$ generated by the operators $l\, S_l = \langle l \rangle\, l^{k-1}$ and T_l, as l ranges over all primes not dividing N.
  - $\mathbb{T}_k(N)$ is commutative, reduced, and free of finite rank over $\mathbb{Z}$. The tensor product $\mathbb{C} \otimes \mathbb{T}_k(N)$ also acts faithfully on $\mathcal{M}_k(N)$.
  - All eigenvalues of the Hecke operators are algebraic integers. The systems of simultaneous eigenvalues for all the Hecke operators on $\mathcal{M}_k(N)$ are closed under$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The space $\mathcal{M}_k(N)$ is the direct sum of simultaneous eigenspaces for Hecke operators.
- We call f a Hecke eigenform if it is an eigenvector for both $l\, S_l$ and T_l for all prime $l \nmid N$. This is equivalent to the existence of a ring hom $\lambda : \mathbb{T}_k(N) \to \mathbb{C}$ such that $T\, f = \lambda(T)\, f$ for all $T \in \mathbb{T}_k(N)$.
  - Hecke eigenforms are "nice" – for example, knowing the first-degree term in the Fourier expansion allows one to determine all the Fourier coefficients recursively.
  - Ex.: Eisenstein series are Hecke eigenforms for $N = 1$. Their eigenvalues are $\lambda(lS_l) = l^{2k-1}$, $\lambda(T_l) = 1 + l^{2k-1}$.
- We primarily care about the Hecke eigenvalues rather than the eigenforms, since the eigenvalues give rise to Galois representations.

2.3 Galois Representations

- Choose a prime p and fix an embedding i_p of $\overline{\mathbb{Q}}$ into $\mathbb{Q}_p$.
- Also fix k and N and write $\mathbb{T}$ for $\mathbb{T}_k(N)$. If λ is a system of Hecke eigenvalues appearing in $\mathcal{M}_k(N)$ then since λ takes values in $\overline{\mathbb{Z}}$ we can compose with our embedding into $\mathbb{Q}_p$ to think of λ as mapping into $\mathbb{Z}_p$. Do this, so that $\lambda : \mathbb{T} \to \overline{\mathbb{Z}_p}$.
- For Σ the primes dividing pN, let $\mathbb{Q}_\Sigma$ be the maximal extension unramified outside Σ and $G_{\mathbb{Q},\Sigma} = \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. For l not in Σ, have $\mathrm{Frob}_l \in G_{\mathbb{Q},\Sigma}$ defined up to conjugacy, and a prime $\mathfrak{L}$ over l with $\mathrm{Frob}_l(x) \equiv x^l \bmod \mathfrak{L}$ for any algebraic integer $x \in \mathbb{Q}_\Sigma$. [Chebotarev implies that the union of the conjugacy classes is dense in $G_{\mathbb{Q},\Sigma}$.]

- Thm: There is a continuous, semisimple representation $\rho_\lambda : G_{\mathbb{Q},\Sigma} \to GL_2\left(\overline{\mathbb{Q}_p}\right)$ uniquely determined up to equivalence by the condition that for each prime l not in Σ, the matrix $\rho_\lambda(\mathrm{Frob}_l)$ has characteristic polynomial given by $X^2 - \lambda(T_l)X + \lambda(lS_l)$.
    - The uniqueness of the representation is not very hard: as noted above, Chebotarev implies that the characteristic polynomials of two such representations agree on a dense set (hence everywhere by continuity), and a semisimple fin-dim rep of a group is uniquely determined up to equivalence by its characteristic polynomials.
    - Existence is more work: part: the proof is due to Deligne ($k > 2$), Eichler-Shimura-Igusa ($k = 2$), and Deligne-Serre ($k = 1$). In the case $k = 2$ this representation really is familiar, as weight-2 modular forms correspond to elliptic curves, and the resulting Galois representation is just the usual one coming from the Tate module. For higher weights the construction involves the cohomology of particular moduli spaces.
    - The polynomial $X^2 - \lambda(T_l)X + \lambda(lS_l)$ is called the lth Hecke polynomial of λ.

3 Crash Course in p-adic Theory

3.1 The p-adic Hecke Algebra

- Let N, k be positive and p not divide N. We define an action of the operators S_l and T_l on the direct sum $\oplus_{i=1}^{k}\mathcal{M}_i(N)$ componentwise (i.e., via the action of the Hecke operators S_l and T_l on each $\mathcal{M}_i(N)$).

- Define $\mathbb{T}_{\leq k}^{(p)}(N)$ to be the $\mathbb{Z}$-subalgebra of endomorphisms of $\oplus_{i=1}^{k}\mathcal{M}_i(N)$ generated by $l\,S_l$ and T_l as l ranges over primes not dividing pN. As before we will drop the (N) from $\mathbb{T}_{\leq k}^{(p)}(N)$.
    - There is an obvious injection of $\mathbb{T}_{\leq k}^{(p)}$ into $\displaystyle\prod_{i=1}^{k}\mathbb{T}_i$ since each operator is determined by its action on each of the direct summands. In fact, the image has finite index.
    - Example: if $N = 1$, $p = 2$, $k = 6$, then M_4 and M_6 are 1-dim (spanned by E_4 and E_6) and the others are trivial. Then $\mathbb{T}_{\leq 6}^{(2)}$ embeds in $\mathbb{Z} \times \mathbb{Z}$ and a little playing with E_4 and E_6 will show that the image is $\{(u, v) : u \equiv v \bmod 12\}$, hence is of index 12.

- Now for $k < k'$ we have an obvious containment of $\oplus_{i=1}^{k}\mathcal{M}_i(N)$ in $\oplus_{i=1}^{k'}\mathcal{M}_i(N)$, hence restricting gives an surjection of $\mathbb{T}_{\leq k'}^{(p)}$ onto $\mathbb{T}_{\leq k}^{(p)}$.

- If we tensor with $\mathbb{Z}_p$ over $\mathbb{Z}$ we get a surjection $\mathbb{T}_{\leq k'}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{T}_{\leq k}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. [We do this because we want something p-adic. And now we will do the only thing one ever does with a diagram like this.]

- We define the p-adic Hecke algebra $\mathbb{T}(N)$ as the inverse limit $\mathbb{T} = \varprojlim_{k} \mathbb{T}_{\leq k}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.
    - As in the classical case we can think of this Hecke algebra as "morally" being generated by lS_l and T_l, interpreted appropriately. [We need to use the "inverse limit versions" of lS_l and T_l, but these are perfectly reasonable – just glue together the images of lS_l and T_l at each stage of the limit.]
    - The ring $\mathbb{T}$ is a p-adically complete, Noetherian $\mathbb{Z}_p$-algebra. [Aka, "fairly nice".]

- We define Hecke eigenforms the same way as in the classical case (i.e., as simultaneous eigenvectors for lS_l and T_l for all l not dividing pN), and as before we will mostly care about the eigenvalues.

3.2 p-adic Galois Representations

- As in the classical case, by a p-adic system of Hecke eigenvalues we mean a ring homomorphism $\xi : \mathbb{T} \to \overline{\mathbb{Z}_p}$, obtained by composing with a map into $\mathbb{Q}_p$.

- Again in analogy with the classical case, one can prove the existence of a continuous, semi-simple representation $\rho_\xi : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ uniquely determined up to equivalence by the condition that for each prime l not dividing pN, the matrix $\rho_\xi(\mathrm{Frob}_l)$ has characteristic polynomial equal to the Hecke polynomial $X^2 - \xi(T_l)X + \xi(lS_l)$.

- ◦ Uniqueness follows from the same argument as before.
- ◦ For existence, if ξ is a system of Hecke eigenvalues which is so-called "p-deprived" then the result is essentially tautological. Then one proves that the systems which are p-deprived are 'dense' in the set of all systems of Hecke eigenvalues, and then uses this result and p-adic interpolation to construct the representation in general.
- ◦ Fundamental Conjecture: If Σ is any finite set of primes containing p, and $\rho : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ is continuous, semisimple, and odd, then $\rho = \rho_\xi$ for some p-adic system of Hecke eigenvalues ξ.

4 Closing Remarks

- You may have noticed I didn't actually define what a p-adic modular form was. This is because every definition I found was far more technically complicated than I felt was useful, or even understandable.
- The interested audience member may refer to Gouvea's book "Arithmetic of p-adic Modular Forms" for the (very) gory technical details.
- For a relatively nice, but deep, introduction of the classical theory of modular forms, I recommend Diamond/Shurman's "A First Course in Modular Forms".
- For a more general and technical discussion of classical results, I liked Miyake's "Modular Forms".