# A Crash Course in Central Simple Algebras

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## 1 Goals

- This is a prep talk for Danny Neftin's talk.
- I aim to cover roughly the following topics:
  - (i) Standard results about central simple algebras, towards a discussion of maximal subfields. (ref: Reiner's book Maximal Orders.)
  - (ii) A discussion of the Brauer group, with a discussion of how cocycles in $H^2$ give $k$-algebras. (ref: Dummit/Foote)
  - (iii) The Albert-Brauer-Hasse-Noether theorem. (ref: Reiner)
  - (iv) The definition of $k$-admissibility and some results thereto, from a paper of Schacher. (ref: Schacher's paper Subfields of Division Rings, I)
- Three reasons why one should care about central simple algebras:
  - The Brauer group plays a rather central (ha!) role in some of the big results in class field theory, which I will briefly mention.
  - Studying maximal orders in central simple algebras is one way of trying to generalize the classical theory of modular forms. (Shimura curves, etc.)
  - Many of the results are really neat.

## 2 The Usual Results About Central Simple Algebras

- Definition: For a field $k$, a central simple $k$-algebra $A$ is a finite-dimensional associative algebra which is simple and whose center is precisely $k$.
- Examples:
  - Any field is a central simple algebra over itself.
  - The quaternions are a real central simple algebra; in fact, they are essentially the only one aside from $\mathbb{R}$ itself.
  - The $n \times n$ matrices over any division ring are a central simple algebra (over the center of that division ring). In fact, these are all the central simple algebras!
- Theorem (Wedderburn): Every left-artinian simple ring is isomorphic to an algebra of matrices over a division ring.
  - This sometimes seems almost too magical a statement, but it's really very concrete. Here is a more explicit version: Let $A$ be a left-artinian simple ring and $I$ be any minimal left ideal of $A$. Then $D = \mathrm{Hom}_A(I, I)$ is a division ring, and $A = \mathrm{Hom}_D(I, I) \cong M_{n\times n}(D^{opp})$, where $n$ is the dimension of the left $D$-module $I$, and $D^{opp}$ is the opposite ring of $D$.
- By Wedderburn's theorem we immediately have that every central simple $k$-algebra is of the form $M_{n\times n}(D)$ for some (unique up to isomorphism) division ring $D$ containing $k$, and some (unique) $n$.

- Definition: We will call $D$ the division ring part of $A$.
- In fact, $Z(A) = \{\alpha I_n : \alpha \in Z(D)\} \cong Z(D)$, so the center of $D$ is also $k$.
- The Frobenius theorem states that the only division rings over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ – a proof is given, of all places, in Silverman 1. Combined with Wedderburn's theorem, we see that every central simple $\mathbb{R}$-algebra is a matrix ring over $\mathbb{R}$ or $\mathbb{H}$ ($\mathbb{C}$ is not possible since its center is not $\mathbb{R}$).
  - For those who like topology: this is related to Hurwitz's theorem classifying which spheres can be fiber products, which is equivalent to asking which normed division algebras exist. (The answer is: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, giving $S^0$, $S^1$, $S^3$, and $S^7$.)
- Every central simple $\mathbb{F}_q$-algebra is isomorphic to $M_{n\times n}(\mathbb{F}_q)$, because a finite division ring is a field by Wedderburn's little theorem.

- Theorem: Let $A$ be a central simple $k$-algebra and $B$ an artinian simple $k$-algebra (not necessarily finite-dimensional). Then $B \otimes_k A$ is an artinian simple algebra with center $Z(B)$.
  - Corollary 1: Let $A$ be a central simple $k$-algebra and $L/k$ a field extension. Then $L \otimes_k A$ is a central simple $L$-algebra.
  - Corollary 2: The tensor product of two central simple $k$-algebras is again a central simple $k$-algebra.

- Theorem: Let $B$ be a simple subring of the central simple $k$-algebra $A$. Define the centralizer of $B$ in $A$, denoted $B'$, to be $B' = \{x \in A : xb = bx \text{ for all } b \in B\}$. Then $B'$ is a simple artinian ring, and $B$ is its centralizer in $A$.
  - Proving this theorem requires a discussion of the double centralizer property, which I won't get into here. But it's neat.
  - Corollary: With notation as above, for $V$ a simple left $A$-module and $D = \mathrm{Hom}_A(V, V)$, then $D \otimes_k B \cong \mathrm{Hom}_{B'}(V, V)$ and $[B : k] \cdot [B' : k] = [A : k]$.
    - The first part is a restatement of the theorem; the second part follows from counting the dimensions of a bunch of related spaces.
  - Corollary: With notation as above, $A \otimes_k B^{opp} \cong M_{r\times r}(B')$ where $r = [B : k]$, and furthermore, $B \otimes_k B' \cong A$ if $B$ has center $k$.
    - This follows, more or less, just by writing everything down as matrix algebras and then counting dimensions.

# 3 Splitting Conditions

- Definition: For $A$ a central simple $k$-algebra, we say that an extension $E$ of $k$ splits $A$ if $E \otimes_k A \cong M_{r\times r}(E)$ for some $r$.
  - Splitting fields always exist; for example, the algebraic closure $\bar{k}$ is always one. This is true because $\bar{k} \otimes_k A$ is a central simple $\bar{k}$-algebra hence is of the form $M_{r\times r}(D')$ for some division ring $D'$ (of finite degree) over $\bar{k}$. But then every element of $D'$ is algebraic over $\bar{k}$, hence actually lies in $\bar{k}$.
  - If $E$ splits $A$, then so does every field containing $E$; just write down the tensor products.

- Theorem: $A$ splits at $L$ if and only if $D$ splits at $L$, where $D$ is the division ring part of $A$.
  - This reduces the question of a central simple algebra's splitting to a simpler one, about a division ring splitting.
  - Proof: Say $A \cong M_{n\times n}(D)$ by Wedderburn.
  - If $D$ splits at $L$, then $L \otimes_k D \cong M_{m\times m}(L)$. Hence we may write $L \otimes_k A \cong L \otimes_k M_{n\times n}(D) \cong M_{n\times n}(L \otimes_k D) \cong M_{n\times n}(M_{m\times m}(L)) \cong M_{mn\times mn}(L)$.
  - Conversely, if $L \otimes_k A \cong M_{r\times r}(L)$ then $M_{r\times r}(L) \cong L \otimes_k M_{n\times n}(D) \cong M_{n\times n}(L \otimes_k D)$. By Wedderburn we know that $L \otimes_k D \cong M_{s\times s}(D')$ for some division ring $D'$, so that $M_{r\times r}(L) \cong M_{ns\times ns}(D')$. But by the uniqueness part of Wedderburn's theorem then forces $D' = L$, so that $L \otimes_k D \cong M_{s\times s}(L)$, as desired.

- Theorem: Let $D$ be a division ring with center $k$, with $[D : k]$ finite. Then every maximal subfield $E$ of $D$ contains $k$ and is a splitting field for $D$, and further, if $m = [E : k]$, then $[D : k] = m^2$ and $E \otimes_k D \cong M_{m\times m}(E)$, where $m$ is called the degree of $D$.
  - Proof: $[D : k]$ is finite so $D$ contains maximal subfields. Clearly any such $E$ must contain $k$, otherwise $E(k)$ would be larger. Now consider the centralizer $E'$ of $E$: obviously $E'$ contains $E$, and in fact we must have equality since for each $x \in E'$, $E(x)$ is a subfield of $D$ containing $E$. So our earlier theorems immediately give $[E : k]^2 = [E' : k] \cdot [E : k] = [D : k]$ and $D \otimes_k E \cong \mathrm{Hom}_E(D, D) \cong M_{r\times r}(E)$ where $r = [D : E] = [E : k]$.
  - If $k$ has positive characteristic, there is a maximal subfield of $D$ which is separable over $k$. This is more of a slog, so I'll skip it.

# 4 The Brauer Group

- Let $L/k$ be an extension of fields, let $D$ be a division ring, and $A$ and $B$ be central simple $k$-algebras.
  - Reminder: if $A \cong M_{r\times r}(D)$ then we refer to $D$ as the division ring part of $A$.
- Definition: We say $A$ and $B$ are similar, denoted $A \sim B$, if their respective division ring parts are $k$-isomorphic. (A $k$-isomorphism is a ring isomorphism which fixes $k$.)
  - Equivalently, by Wedderburn's theorem, there exist integers $r$ and $s$ so that $A \otimes_k M_{r\times r}(k) \cong B \otimes_k M_{s\times s}(K)$.
  - Denote the equivalence class of $A$ under $\sim$ by $[A]$.
- Theorem: The classes of central simple $k$-algebras form an abelian group $B(k)$, called the Brauer group of k, with multiplication given by tensor product, with identity $[k]$ and with $[A]^{-1} = [A^{opp}]$.
  - Proof: From before we know that the tensor product $A \otimes_k B$ is also a central simple $k$-algebra, so we have a well-defined multiplication of classes $[A][B] = [A \otimes_k B]$.
  - This operation is obviously associative, commutative and has identity $[k]$, so we need only check that $[A][A^{opp}] = [k]$.
  - From before we also know that $A \otimes_k B^{opp} \cong M_{r\times r}(B')$, so by taking $A = B$, so that $B' = k$, we obtain $[A][A^{opp}] = [M_{r\times r}(k)] = [k]$.
- Proposition: For $k \subset L$, we have a group homomorphism $B(k) \to B(L)$ via $[A] \mapsto [L \otimes_k A]$ for $[A] \in B(k)$.
- Definition: Define $B(L/k)$ to be the kernel of the map $B(k) \to B(L)$; then $[A] \in B(L/k)$ iff $L \otimes_k A \cong M_r(L)$ for some $r$. (Recall that we say that $L$ splits $A$.)

# 5 $H^2$ and the Crossed Product Construction

- Definition: For any group $G$ and $G$-module $A$, a 2-cocycle is a function $f : G \times G \to A$ satisfying the cocycle condition $f(g, h) + f(gh, k) = g \cdot f(h, k) + f(g, hk)$ for all $g, h, k \in G$.
  - Equivalently, a 2-cocycle is determined by a collection of elements $a_{g,h}$ in $A$ (called a factor set) with the property that $a_{g,h} + a_{gh,k} = g \cdot a_{h,k} + a_{g,hk}$, and the 2-cocycle $f$ is the function sending $(g, h) \mapsto a_{g,h}$.
  - The multiplicative form of this relation is $a_{\sigma,\tau}a_{\sigma\tau,\rho} = \sigma(a_{\tau,\rho})a_{\sigma,\tau\rho}$, for $\sigma, \tau, \rho \in G$.
- Definition: A 2-cochain $f$ is a coboundary if there is a function $f_1 : G \to A$ such that $f(g, h) = g \cdot f_1(h) - f_1(gh) + f_1(g)$ for all $g, h \in G$.
  - The cohomology group $H^2(G, A)$ is the group of 2-cocycles modulo coboundaries, as with every cohomology group ever.

- One reason that $H^2$ is interesting (in general group cohomology) is that the cohomology classes correspond bijectively to equivalence classes of extensions of $G$ by $A$; namely, to short exact sequences $1 \to A \to E \to G \to 1$, where extensions are equivalent if there is an isomorphism of $E$ which makes this diagram commute: $\begin{array}{ccccccccc} 1 & \to & A & \to & E & \to & G & \to 1 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 1 & \to & A & \to & E & \to & G & \to 1 \end{array}$, where the maps from $A \to A$ and $G \to G$ are the identity. Split extensions correspond to the trivial cohomology class.

- Definition: A 2-cocycle is called a normalized 2-cocycle if $f(g,1) = 0 = f(1,g)$ for all $g \in G$.
  - One may verify that each 2-cocycle lies in the same cohomology class as a normalized 2-cocycle: explicitly, if $f'$ is the 2-coboundary whose $f_1$ is identically $f(1,1)$ (which is to say $f'(g,h) = g \cdot f(1,1)$) then one can check that $f - f'$ is normalized.
  - So we may as well just deal with normalized 2-cocycles when talking about elements of the cohomology group, since it makes life easier.

- If $L/k$ is a finite Galois extension of fields with Galois group $G = \mathrm{Gal}(L/k)$ then we can use the normalized 2-cocycles in $Z^2(G, L^\times)$ to construct central simple $k$-algebras using the crossed product construction. Here is the construction:
  - Suppose $f = \{a_{\sigma,\tau}\}_{\sigma,\tau \in G}$ is a normalized 2-cocycle in $Z^2(G, L^\times)$ and let $B_f$ be the vector space over $L$ having basis $u_\sigma$ for $\sigma \in G$.
  - Thus elements of $B_f$ are sums of the form $\sum_{\sigma \in G} \alpha_\sigma u_\sigma$ where the $\alpha_\sigma$ lie in $L$.
  - Define a multiplication on $B_f$ by $u_\sigma \alpha = \alpha(\sigma) u_\sigma$, and $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$, for $\alpha \in L$ and $\sigma, \tau \in G$.

- Theorem: $B_f$ is a central simple $k$-algebra split at $L$, and, furthermore, choosing a different cocycle in the same cohomology class produces a $k$-isomorphic $k$-algebra.
  - We need to check associativity, find an identity, check that the center is $k$, and show that it is simple. We will also verify that $L$ is maximal and that the choice of cocycle does not matter.
  - Associativity: One can compute from this definition that $(u_\sigma u_\tau) u_\rho = a_{\sigma,\tau} a_{\sigma\tau,\rho} u_{\sigma\tau\rho}$ and $u_\sigma(u_\tau u_\rho) = \sigma(a_{\tau,\rho}) a_{\sigma,\tau\rho} u_{\sigma\tau\rho}$. But $a_{\sigma,\tau} a_{\sigma\tau,\rho} = \sigma(a_{\tau,\rho}) a_{\sigma,\tau\rho}$ is precisely the multiplicative form of the cocycle condition, so the multiplication is associative.
  - Identity: Since we assumed the cocycle was normalized, we have $a_{1,\sigma} = a_{\sigma,1} = 1$ for all $\sigma \in G$, so $u_1$ is an identity in $G$.
  - Center is $k$: If $x = \sum_{\sigma \in G} \alpha_\sigma u_\sigma$ is in the center, then $x\beta = \beta x$ for all $\beta \in L$ shows that $\sigma(\beta) = \beta$ if $\alpha_\sigma \neq 0$. But since there is an element of $L$ not fixed by $\sigma$ (for any $\sigma \neq 1$), we get $\alpha_\sigma = 0$ for all $\sigma \neq 1$. Hence $x = \alpha_1 u_1$; then $x u_\tau = u_\tau x$ iff $\tau(\alpha_1) = \alpha_1$ for all $\tau \in G$, which just says that $\alpha_1$ is fixed by the entire Galois group (i.e., is in $k$).
  - Simple: Let $I$ be a nonzero ideal and take any $x = \alpha_{\sigma_1} u_{\sigma_1} + \cdots + \alpha_{\sigma_m} u_{\sigma_m}$ in $I$ with the minimal number of terms. If $m > 1$ then there is an element $\beta \in L^\times$ with $\sigma_m(\beta) \neq \sigma_{m-1}(\beta)$. But then $x - \sigma_m(\beta)\, x\, \beta^{-1}$ is in $I$, but has zero $u_{\sigma_m}$ term but nonzero $u_{\sigma_{m-1}}$ term. Hence $m = 1$ and $x = \alpha u_\sigma$, and this element is a unit with inverse $\sigma^{-1}(\alpha^{-1}) u_{\sigma^{-1}}$.
  - Cohomology representative does not matter: If $f' = \{a'_{\sigma,\tau}\}$ is a different representative of the cohomology class of $f$, then the multiplicative form of the coboundary condition says that there exist elements $b_\sigma \in L^\times$ with $a'_{\sigma,\tau} = a_{\sigma,\tau}(\sigma(b_\tau) b_{\sigma\tau}^{-1} b_\sigma)$. Let $\varphi$ be the $L$-vector space homomorphism defined by $\varphi(u'_\sigma) = b_\sigma u_\sigma$: then one can push symbols to see that $\varphi(u'_\sigma u'_\tau) = \varphi(u'_\sigma)\varphi(u'_\tau)$. Hence $\varphi$ is a $k$-algebra isomorphism of $B_f$ and $B_{f'}$.
  - Split at $L$: Upon identifying $L$ with the elements $\alpha\, u_1$ in $B_f$, we see that $B_f$ is a $k$-algebra containing $L$, and has $[B_f : k] = [L : k]^2$. By our results earlier on central simple algebras, this tells us that $L$ is a maximal subfield of $B_f$. Applying the theorem about $A \otimes_k B^{opp} \cong M_{r \times r}(B')$ with $A = B = B' = B^{opp} = L$ shows that $B_f$ splits at $L$.

- The above theorem tells us that $B(L/k)$ and $H^2(G, L^\times)$ are two groups which share the same elements. We should expect that they're actually isomorphic as groups, which indeed they are, but this requires a little more work.
  - If we start with the trivial cohomology class, we should end up with the trivial element of the Brauer group – namely, $M_{n\times n}(k)$ – and indeed, we do, although it requires some checking.
  - Similarly, the addition in $H^2$ corresponds to tensor product; this takes a fair bit of additional effort.
  - Remark: This result shows that every division ring $D$ with center $k$ such that $[D : k]$ is finite, is similar to some crossed product algebra. However, there exist (infinite-dimensional) division rings which are not isomorphic to crossed-product algebras.

# 6 Albert-Brauer-Hasse-Noether

- Theorem (Albert-Brauer-Hasse-Noether): If $A$ is a central simple $k$-algebra, then $A \sim k$ if and only if $A_p \sim k_p$ for each prime $p$ of $k$.
  - The forward direction is obvious (localization plays nice with matrices); the reverse direction is hard. I won't go into the proof, aside from mentioning that it uses the Hasse Norm Theorem.
  - This is a "Hasse principle" sort of theorem: it tells us that if a central simple $k$-algebra splits at each prime $p$, then the algebra splits globally.
  - For each prime $\mathfrak{p}$ of $k$, there is a homomorphism $B(k) \to B(k_\mathfrak{p})$ defined by $[A] \mapsto [k_\mathfrak{p} \otimes_k A]$. For $m_\mathfrak{p}$ the local index of $A$ at $\mathfrak{p}$ (which I won't define here), we have $m_\mathfrak{p} = 1$ hence $[A_\mathfrak{p}] = 1$ for all but finitely many $\mathfrak{p}$. So we have a well-defined homomorphism $B(k) \to \sum_\mathfrak{p} B(k_\mathfrak{p})$; the Albert-Brauer-Hasse-Noether theorem is precisely the statement that this map is injective.
- A stronger result, due to Hasse, fits this map into the following exact sequence: $1 \to B(k) \to \sum_\mathfrak{p} B(k_\mathfrak{p}) \stackrel{\text{inv}}{\to} \mathbb{Q}/\mathbb{Z} \to 0$, where inv denotes the Hasse invariant map.
  - Neukirch proves the exactness of this sequence first and then deduces the above results as corollaries.
  - The usual method of doing it this way is to prove that $1 \to H^2(G_{L/k}, L^\times) \to \sum_\mathfrak{p} H^2(G_{L_\mathfrak{p}/k_\mathfrak{p}}, L_\mathfrak{p}^\times) \stackrel{\text{inv}}{\to} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0$, for all finite cyclic extensions $L/k$ of degree $n$.
  - Then apply the relation between $H^2$ and the Brauer groups (namely, $H^2(G_{L/k}, L^*) \cong B(L/k)$, and the same inside the direct sum) and then show that $B(k) = \bigcup_L B(L/k)$ where the union is taken over finite cyclic extensions of $k$.
- Corollary: For $A$ a central simple $K$-algebra with local indices $\{m_\mathfrak{p}\}$, then the order of $[A]$ in $B(k)$ is $\text{lcm}(m_\mathfrak{p})$.
  - Proof: We have $[A]^t = 1$ in $B(k)$ iff $[A_p]^t = 1$ in $B(k_p)$ for each $p$, but by the Hasse invariant we know that the order of $[A_p]$ in $B(k_p)$ is $m_p$.
- One can use the Grunwald-Wang theorem in concert with Albert-Brauer-Hasse-Noether to prove the following result: if $k$ is a global field then the order of $[A]$ in the Brauer group is equal to $\text{index}[A] = \sqrt{[A : k]}$.
- Another corollary of Albert-Brauer-Hasse-Noether is the following: For $k$ a global field and $D$ a division ring with center $k$, there exists a maximal subfield $E$ of $D$ which is a cyclic extension of $k$.

# 7 Schacher's paper

- Definition: If $L/k$ is a finite extension of fields, then $L$ is $k$-adequate if there is a division ring $D$ with center $k$ containing $L$ as a maximal commutative subfield; otherwise $L$ is $k$-deficient.
- Definition: A finite group $G$ is $k$-admissible if there is a Galois extension $L/k$ with Galois group $G$, and $L$ is $k$-adequate.
- A $k$-division ring is a division ring $D$ finite-dimensional over its center $k$. From earlier results we know that $[D : k] = n^2$ where $n = [E : k]$ is called the degree of $D$, and $E$ is (any) maximal subfield.

- Let $m$ be the order of $[D]$ in $B(k)$. We call $k$ stable if $m = n$ for every $k$-division ring $D$; we just mentioned that Grunwald-Wang plus Albert-Brauer-Hasse-Noether shows that global fields are stable.

- Also from Albert-Brauer-Hasse-Noether, we know that $D$ has a maximal subfield (in fact, the proof shows there are infinitely many nonisomorphic choices) which is cyclic over $k$. However, the theorem says nothing about what other maximal subfields are possible.

- Prop 2.1: If $k$ is stable, then $L$ is $k$-adequate iff $B(L/k)$ has an element of order $[L : k]$.

  - Proof: definition chase.

- Prop 2.2: If $k$ is stable, then $L$ is $k$-adequate iff $L$ is contained in a $k$-division ring.

  - In other words, for stable fields, the maximality condition comes for free.
  - Proof: If $k \subset L \subset D$, let $M$ be a maximal subfield of $D$ containing $L$. Then use the exact sequence $0 \to B(L/k) \to B(M/k) \to B(M/L)$ to get an element of the proper order in $B(L/k)$ from an element in $B(M/k)$.

- Now assume $k$ is a global field and $L$ is a finite Galois extension of $k$ with $G = \text{Gal}(L/k)$ and $|G| = n$. We know that $L$ is $k$-adequate iff $H^2(G, L^\times)$ has an element of order $n$; since this group is abelian we need only determine if it has an element of order $p_i^{l_i}$ for each prime power $p_i^{l_i}$ in the factorization of $n$.

- Also recall we have the exact sequence $1 \to H^2(G_{L/k}, L^\times) \to \sum_{\mathfrak{p}} H^2(G_{L_\mathfrak{p}/k_\mathfrak{p}}, L_\mathfrak{p}^\times) \overset{\text{inv}}{\to} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0$.

- Prop 2.5: With notation as above, for $p$ a prime and $r$ an integer, $H^2(G, L^\times)$ contains an element of order $p^r$ if and only if $n_\mathfrak{q} = [L_\mathfrak{q} : k_\mathfrak{q}]$ is divisible by $p^r$ for two different primes $\mathfrak{q}$ of $k$.

  - Proof: Suppose $a \in H^2$ has order $p^r$. Write $a = a_{q_1} + \cdots + a_{q_r}$ for $a_{q_i} \in H^2(G_{L_\mathfrak{q}/k_\mathfrak{q}}, L_\mathfrak{q}^\times)$ for some primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of $k$. Then one of the $a_{q_i}$ must have order divisible by $p^r$ since the order of $a$ is $p^r$ – but the sum of the invariants being 0 forces at least one other of the $a_{q_i}$ to be divisible by $p^r$ as well. Then for these two, clearly $n_{q_i} = [L_{q_i} : k_{q_i}]$ is also divisible by $p^r$.
  - Conversely, suppose that $n_{q_1}$ and $n_{q_2}$ are divisible by $p^r$. Then we can find $a_{q_1} \in H^2(G_{L_{\mathfrak{q}_1}/k_{\mathfrak{q}_1}}, L_{\mathfrak{p}_1}^\times)$ and $a_{q_2} \in H^2(G_{L_{\mathfrak{q}_2}/k_{\mathfrak{q}_2}}, L_{\mathfrak{q}_2}^\times)$ with $a_{q_1}$ having Hasse invariant $1/p^r$ and $a_{q_2}$ having invariant $-1/p^r$. Then $a_{q_1} + a_{q_2}$ has order $p^r$ in $H^2(G, L^\times)$.

- Prop 2.6: With notation as above, if $p^r$ is the highest power of $p$ dividing $n$, then $H^2(G, L^\times)$ has an element of order $p^r$ iff $G_q = \text{Gal}(L_q/k_q)$ contains a $p$-Sylow subgroup of $G$ for two different primes $\mathfrak{q}$ of $k$.

  - This is just a restatement of 2.5, using the fact that $G_q$ is a subgroup of $G$.

- Example (non-adequate extension): Let $k = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_8)$. Then $[L : k] = 4$ with Galois group the Klein 4-group. $L$ is unramified at odd primes, hence $G_p$ is either $\mathbb{Z}/2\mathbb{Z}$ or 0 for $p > 2$, and $G_2$ is the Klein 4-group. Hence by Prop 2.5, $H^2$ has no elements of order 4, so $L$ is $\mathbb{Q}$-deficient.

  - We can rephrase this result (using the equivalent criterion for adequacy) as: any division ring with center $\mathbb{Q}$ containing a root of $x^4 + 1$ is infinite-dimensional.

- As one might expect, it seems like it would not be too hard to work out these computations in examples with reasonably nice Galois groups – things like $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ or $\mathbb{Q}(\zeta_n)$ – to see which ones are $\mathbb{Q}$-adequate.

- Theorem (Schacher): If $G$ is $\mathbb{Q}$-admissible, then every Sylow subgroup of $G$ is metacyclic (i.e., is a cyclic extension of a cyclic group). For abelian groups, the converse also holds.

  - One might guess that, based on some examples, every $\mathbb{Q}$-admissible group is solvable, but this is not true: $S_5$ is also $\mathbb{Q}$-admissible.
  - If we allow ourselves to raise the base field away from $\mathbb{Q}$, we can get other groups. In fact....

- Theorem (Schacher): For any finite group $G$, there exists a number field $k$ such that $G$ is $k$-admissible.

- The situation does not carry over to function fields: many groups are not admissible over any global field of nonzero characteristic. For example....

- Theorem (Schacher): For $k$ a global field of characteristic $p$, then if $G$ is $k$-admissible then every $q$-Sylow subgroup of $G$ is metacyclic for $q \neq p$.
  - In particular, $S_9$ is not admissible over any function field, as both its 2-Sylow and 3-Sylow subgroups are not metacyclic.