# 1 Intro and History

- Motivating example: $1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + - - + \cdots = \frac{\ln(1+\sqrt{2})}{\sqrt{2}}$. This is $L(1, \chi)$ for the nontrivial $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$ with $\chi(-1) = 1$.
  - The question is, broadly, why is this sum so nice?
  - No $\pi$s, unlike if the signs just alternated. Just the natural log of a nice algebraic number – a unit, in fact – times another algebraic number.
- Basic motivating idea: The analytic class number formula relates algebraic information to an L-function, for quadratic extensions. Can we say something in higher degree, provided our fields are nice enough?
  - Stark, in the 1970s, proved some theoretical results suggesting various results in the "yes" column, which led him to make several conjectures.
  - First examples of Stark: looking at cyclotomic fields suggests units are involved.
  - Brumer then reinterpreted Stickelberger's theorem on the factorization of the Gauss sum, in terms of an annihilator of a class group. This led to the Brumer-Stark conjecture.
  - Stark verified some specific cases by computations (over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-D})$, and some other cases).
  - Tate, in 1980s, added his take and put things in a broader context.
  - Many others, 1990s-present [D. Dummit, K. Rubin, S. Dasgupta, M. Greenberg, H. Cohen, et al.] have added data also refining various conjectures and suggesting generalizations.
- We will restrict attention to the "rank 1 abelian case", in which the extension is abelian and the L-function vanishes to order 1, because these are 'nicer' and there is a more refined version of the general conjecture. [Hence, "Stark's conjectures", plural.]
  - This case is also the topic of the Arizona Winter School project.
- Why do we care?
  - If we could prove the conjectures, we could explicitly construct various class fields; in particular the Hilbert class field in many cases. This is something we really like – it would solve Hilbert's 12th problem. I'll give examples.
  - Results are very conducive to computation, for single cases. So it's not too hard to get our hands on examples (these days).

# 2 Ingredients

- Let $K/k$ be a finite abelian extension of number fields, with $G = \mathrm{Gal}(K/k)$. Let $S$ be a finite set of places in $k$, which minimally contains (i) all Archimedean places, and (ii) all places which ramify in $K$.
  - We recall the definition of an $S$-unit: $x \in K$ is an $S$-unit if the fractional ideal $(x)$ is a product of primes in $S$.
  - Dirichlet's $S$-unit theorem states that the group of $S$-units is finitely generated and has rank equal to $r + \#(S)$, where $r$ is the rank of the (usual) unit group.
- Definition: The (imprimitive) L-series for a character $\chi$ (relative to $S$) of $G$ is

$$L_S(s, \chi) = \sum_{(\mathfrak{a},S)=1} \frac{\chi(\sigma_\mathfrak{a})}{\mathrm{Nm}(\mathfrak{a})^s} = \prod_{(\mathfrak{p},S)=1} \left(1 - \frac{\chi(\sigma_\mathfrak{p})}{\mathrm{Nm}(\mathfrak{p})^s}\right)^{-1}$$

  Note that the sum is over all integral ideals relatively prime to the primes in $S$, while the product is over all primes $\mathfrak{p} \in k$ not in $S$, and $\sigma_\mathfrak{p}$ is the Frobenius symbol for the extension $K/k$.

- Remark: The order of vanishing of $L_s(s,\chi)$ at $s=0$ is denoted $r(\chi)$; we also call this the rank of $\chi$. If $\chi$ is the trivial character then $r(\chi)=|S|-1$, and if $\chi$ is not the trivial character then $r(\chi)=\#\{v\in S : \chi(D_v)=1\}$ where $D_v$ is the decomposition group for $v$.
- Example (Quadratic cases): If $K=\mathbb{Q}(\sqrt{D})$ then for $\chi_0$ the trivial character and $\chi_D$ not trivial and $\epsilon_0$ a fundamental unit for $D>0$, we have

$$\begin{aligned} L_S(s,\chi_0) &= -\frac{1}{2}\cdot\left(\prod_{\text{finite } p\in S}\ln(p)\right)\cdot s^{|S|-1}+\dots \\ L_S(s,\chi_D) &=^{D<0} h_K\cdot 2^{\#(\text{inert } p\in S)}\cdot\left(\prod_{\text{split } p\in S}\ln(p)\right)\cdot s^{\#(\text{split } p\in S)}+\dots \\ &=^{D>0} h_K\cdot\ln(\epsilon_0)\cdot 2^{\#(\text{inert } p\in S)}\cdot\left(\prod_{\text{split } p\in S}\ln(p)\right)\cdot s^{1+\#(\text{split } p\in S)}+\dots \end{aligned}$$

- Definition: The partial zeta-function is $\boxed{\zeta_S(s,\sigma)=\sum_{\substack{\mathfrak{a}\text{ integral}\\(\mathfrak{a},S)=1\\ \sigma_a=\sigma}}\frac{1}{(\mathrm{Nm}(\mathfrak{a}))^s}}$, where $\sigma$ is any element of $G$. Note that the sum is over all integral ideals relatively prime to primes in $S$, such that the Frobenius symbol $\sigma_{\mathfrak{a}}$ equals $\sigma$.
  - Remark: The partial zeta functions and the L-series are Fourier inverses. Explicitly: $L_S(s,\chi)=\sum_{\sigma\in G}\chi(\sigma)\cdot\zeta_S(s,\sigma)$.
  - Theorem (Siegel): The value $\zeta_S(0,\sigma)$ is a rational number.

# 3 The Rank-One Abelian Stark Conjecture

- The result was originally conjectured in the 1970s by Stark, and a great deal of numerical evidence has been amassed since then.
- So there are several different conjectures which are all called the "rank-1 abelian Stark conjecture". The one I give here is stronger than the original version actually conjectured by Stark; it is due to Tate.

## 3.1 Statement

- The Rank-One Abelian Stark Conjecture: If $K/k$ is abelian, $S$ is a finite set of places containing all Archimedean places and all places ramifying in $K/k$, and $L_S(s,\chi)$ vanishes to order 1 at $s=0$. Also suppose that $S$ contains at least 2 places, including one place $v$ which splits completely in $K$. Then there exists an $S$-unit $\epsilon$ (which we call a Stark unit), with the following properties:
  - If $S$ contains at least 3 places, then $|\epsilon|_{w'}=1$ for all places $w'$ of $K$ not dividing the place $v$ of $k$. In particular, $\epsilon$ is a $v$-unit in $K$. If $S=\{v,v'\}$ then $|\epsilon|_{\sigma w'}=|\epsilon|_{w'}$ for all $\sigma\in G$ and all places $w'$ of $K$ dividing $v'$.
  - For all $\sigma\in G$, $\boxed{\log|\epsilon^\sigma|_w=-e\cdot\zeta_S'(0,\sigma)}$, where $e$ is the number of roots of unity in $K$, $w$ is a prime of $K$ lying above the prime $v$ of $k$ that splits completely in $K$, and the absolute values are normalized. Equivalently, $\boxed{L_S'(0,\chi)=-\frac{1}{e}\cdot\sum_{\sigma\in G}\chi(\sigma)\cdot\log|\epsilon^\sigma|_w}$, for all characters $\chi$ of $G$.
  - The extension $K(\epsilon^{1/e})$ is an abelian extension of $k$.
- Stark proved this conjecture in several cases, including the case $k=\mathbb{Q}$ and $k=\mathbb{Q}(\sqrt{-D})$ .
  - For $k=\mathbb{Q}$, the Stark unit is the so-called "circular unit". Follows from classical theory; classical theory of cyclotomic fields, and class number formula.

- For $k = \mathbb{Q}(\sqrt{-D})$, the Stark unit is an "elliptic unit". This is much harder. [Morally follows from the Kronecker limit formula.]

- Any complex Archimedean prime of $k$ splits completely in $K$, and $L'_S(0,\chi) = 0$ for all $\chi$ so the conjecture is (essentially) trivially true with $\epsilon = 1$ when there is more than one complex Archimedean prime.
  - So the 'interesting cases' of the abelian rank-one conjecture are
    - Type I fields: $k$ has signature $(n, 0)$ – i.e., $k$ is totally real.
    - Type II fields: $k$ has signature $(n-2, 1)$ and $v$ is the complex Archimedean place in $k$.
    - We generally will further split Type I fields, depending on whether $v$ is an Archimedean place or a finite place.

### 3.2 Reformulation of the Conjecture [Representations]

- For $U_S$ the group of $S$-units of $K$, let $Y$ be the free abelian group on the primes $S_K$ in $K$ over $S$, and $X$ be the elements of trace 0 in $Y$. In other words, $X = \left\{ \sum_{w \in S_K} n_w w \ : \ \sum_{w \in S_K} n_w = 0 \right\}$.
- Both $U_S$ and $X$ are $\mathbb{Z}[G]$-modules. The logarithm map $\lambda_K : \epsilon \to \sum_{w \in S_K} \log |\epsilon|_w w$ yields an isomorphism of $U_S \otimes_{\mathbb{Z}} \mathbb{C}$ with $X \otimes_{\mathbb{Z}} \mathbb{C}$, as $\mathbb{C}[G]$-modules. [This is just the Dirichlet $S$-unit theorem.]
- So in particular, the $\chi$-isotypical components on both sides are equal: $(U_S \otimes_{\mathbb{Z}} \mathbb{C})^{(\chi)} \cong (X \otimes_{\mathbb{Z}} \mathbb{C})^{(\chi)}$.
- Since an isomorphism of representations of $G$ over $\mathbb{C}$ 'morally' only cares about what happens over $\mathbb{Q}$, we know that the same statements hold when we tensor with $\mathbb{Q}$ instead of $\mathbb{C}$.
- In this guise, Stark's Conjecture states that the $\mathbb{Q}$-isomorphism of these two spaces is (essentially) given by the leading coefficients of the Taylor expansions of the $L_S(s,\chi)$ at $s = 0$.

### 3.3 Proof of conjecture for $k = \mathbb{Q}$, $v$ Archimedean.

- Here I give the proof of the conjecture in the case where the base field is $\mathbb{Q}$, and the place $v$ which splits completely is Archimedean. This is from Tate's book.
- The result in general essentially follows from the cyclotomic case.
- Let $m \geq 3$ be odd or divisible by 4, and $\zeta_m = e^{2\pi i/m}$ be a primitive $m$th root of unity.
- Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_m)^+$, the maximal real subfield of $\mathbb{Q}(\zeta_m)$, and $S = \{p_\infty, \text{primes dividing } m\}$.
- We know $G \cong (\mathbb{Z}/m)^\times / \{\pm 1\}$, canonically. Let $\sigma_a$ be the restriction to $K$ of the automorphism of $\mathbb{Q}(\zeta_m)$ that sends $\zeta \to \zeta^a$; observe $\sigma_a = \sigma_{-a}$.
- Let $\epsilon = (1-\zeta_m)\cdot(1-\zeta_m^{-1}) = 2 - 2\cos(2\pi/m)$. We claim that $\epsilon$ is the desired Stark unit.
  - It is basic that if $|S| \geq 3$ then $\epsilon$ is a unit, and if $|S| = 2$ (ie., $m = p^k$) then $\epsilon$ is an $S$-unit. Thus $\epsilon$ satisfies part (i).
- Observe $\epsilon^\sigma = (1-\zeta_m^a)(1-\zeta_m^{-a}) = 2 - 2\cos(2\pi a/m)$.
- The partial zeta functions in this case are basically just the classical Hurwitz zeta functions: $\zeta_S(s,\sigma_a) = \sum_{n \equiv \pm a \text{ mod m}, n>0} |n|^{-s} = \sum_{n \equiv a \text{ mod } m} |n|^{-s}$.
  - It is a standard analytic number theory problem to compute the series for these partial zeta functions.
  - The answer is $\zeta'_S(0,\sigma_a) = -\frac{1}{2}\log\left[2 - 2\cos(2\pi a/m)\right] = \frac{1}{2}\log(\epsilon^{\sigma_a})$.
  - Since $K$ is a subfield of $\mathbb{R}$, we have $e = 2$. This proves part (ii) of the Conjecture.

- We also observe that $\epsilon = 2 - \zeta_m - \zeta_m^{-1} = \left[\zeta_4 \cdot (\zeta_{2m} - \zeta_{2m})^{-1}\right]^2$, hence that $\epsilon$ is the square of an $S$-unit inside $K_{4m} = \mathbb{Q}(\zeta_{4m})$. Hence in particular, $K(\sqrt{\epsilon})$ is abelian over $\mathbb{Q}$; this proves part (iii) of the Conjecture.

### 3.4 Interesting example: $K = \mathbb{Q}(\sqrt{p})$.

- Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \bmod 4$. Take $S = \{\mathfrak{p}_\infty, p\}$ and $v$ to be the Archimedean place.
- One can compute that (with $\epsilon_0$ the fundamental unit) the Stark unit $\epsilon$ would satisfy

$$\begin{aligned} L_S'(0,\chi_0) &= -\frac{1}{2}\cdot\log(p) = -\frac{1}{2}\left[\log(\epsilon)+\log(\epsilon^\sigma)\right] \\ L_s'(0,\chi_p) &= h_p\cdot\log(\epsilon_0) = -\frac{1}{2}\left[\log(\epsilon)-\log(\epsilon^\sigma)\right] \end{aligned}$$

- In particular, we see $\epsilon = \sqrt{p}\cdot\epsilon_0^{-h_p}$ (so $\epsilon$ actually exists).
- But also since $\epsilon > 0$ hence $\epsilon^\sigma > 0$ [because $K(\sqrt{\epsilon})$ is abelian over $\mathbb{Q}$ by Stark, plus some thinking].
- Then the inequality $0 < \mathrm{Nm}_{K/\mathbb{Q}}(\epsilon) = \dfrac{-p}{\mathrm{Nm}_{K/\mathbb{Q}}(\epsilon_0)^{h_p}}$ forces $h_p$ to be odd and for $\mathrm{Nm}_{K/\mathbb{Q}}(\epsilon_0) = -1$.
  - Both interesting facts that follow immediately from Stark!
- Remark: There are lots of other examples like this – e.g., one can do a similar calculation for $K = \mathbb{Q}(\sqrt{p},\sqrt{q})$ to show that 2 divides $h_{pq}$ and $\mathrm{Nm}_{K/\mathbb{Q}}(\epsilon_{pq}) = -1$.

### 3.5 Class Fields for Type I fields, with $v$ Archimedean

- So I mentioned that Stark being true would allow us to find class fields. How does this work?
- General idea: start in the base field $k$, then find some nice extension of the Hilbert class field $H$ to which Stark's conjectures apply, and then hope we can figure out $H$ from this.
- Easiest possible thing to try: take $L_i = H(\sqrt{\alpha})$ for some $\alpha \in k$.
  - First, need $L_i/k$ to be abelian. This is not too hard to manage; there are various easy criteria for such $\alpha$.
  - Second, need to arrange this business about having one real place split and the rest become complex. This part is also doable – it's enough to choose $\alpha$ to be positive at one real place, and negative at all the others.
  - Now apply Stark to see that there should exist $\epsilon \in L$, whose logs of conjugates give the $L$-series – $L_S'(0,\chi) = -\dfrac{1}{e}\cdot\sum_{\sigma\in G}\chi(\sigma)\cdot\log|\epsilon^\sigma|_w$.
  - So it's enough to compute the $L$-series for all the characters $\chi$, and then Fourier-invert and exponentiate, to get $\epsilon$ and its conjugates.
    - ∗ Dummit-Sands-Tangedal gave an algorithm for computing these $L$-series to arbitrary accuracy, which is implemented in PARI-GP.
  - Once we have an approximate value for $\epsilon$, we can use numerical magic to find what $\epsilon$'s minimal polynomial should be, and hence get an exact value for $\alpha$, we hope! [Approximate to 20 decimal places and let a computer hack away.]
  - Now take the trace, which will lie in $H$ – $\mathrm{tr}_{L/H}(\epsilon) = \epsilon + \bar{\epsilon} = \epsilon + \epsilon^{-1}$. In many cases only one Stark unit is needed to generate $H$ [one can prove this for certain cubic extensions, for example], but in general we may need to compute several $\epsilon_i$ in order to generate $H$.
- In fact, this rather sketchy method works for $k$ totally real, as justified by the following theorem of Roblot (which allows one to compute real ray class fields):
  - Theorem: Suppose $k$ is a totally real field and $v$ is an Archimedean prime in $k$. Also let $L$ be any finite abelian extension in which $v$ splits completely. Then there exist Stark units $\epsilon_1,\ldots,\epsilon_m$ such that $L = \mathbb{Q}(\epsilon_1+\epsilon_1^{-1}+\cdots+\epsilon_m+\epsilon_m^{-1})$.
- Example: Take $\mathbb{Q}(\sqrt{82})$. It is not pure joy (but not impossible) to see that the class number is 4, so the Hilbert class field is of degree 8 over $\mathbb{Q}$. Applying this method yields a Stark unit $\epsilon$, which (upon using a computer to find a smaller generator) shows that the Hilbert class field is generated by a root of the polynomial $x^8 - 4x^7 - 14x^6 + 56x^5 + 49x^4 - 196x^3 + 28x^2 + 80x - 25$.

### 3.6 Type I fields with $v$ finite: Brumer-Stark

- The rank-one abelian Stark conjecture for the case where $K$ is a totally complex abelian extension of a totally real field $k$, $v$ is a finite prime, and $S = \{v, p_{\text{infinite}}, p_{\text{ramified}}\}$ has been refined a little more (due to Brumer) and is called the Brumer-Stark conjecture.

- Definition: The generalized Stickelberger element is $\theta_{S,G} = \sum_{\sigma \in G} \zeta_S(0, \sigma) \cdot \sigma^{-1} \in \mathbb{Q}[G]$.

  - The element gets its name from the Stickelberger element, which arises in Stickelberger's theorem on the prime factorization of Gauss sums.
  - The element $\theta_{S,G}$ of the group ring has various functorial properties.
  - By a theorem of Deligne and Ribet, there is a uniform bound on the denominators of this element: explicitly, $e\theta_{S,G} = e \sum_{\sigma \in G} \zeta_S(0, \sigma)\sigma^{-1}$ has integral coefficients.

- Brumer-Stark Conjecture: With $K, k, S$ as above, for every fractional ideal $\mathfrak{B}$ of $K$ there exists a $\beta \in K$ with absolute value 1 at all Archimedean places (an 'anti-unit') such that $\mathfrak{B}^{e\theta} = (\beta)$, and $K(\beta^{1/e})$ is abelian over $k$.

- Popescu and Greither have announced that they have proven (though not published yet) a large chunk of the Brumer-Stark conjecture.

### 3.7 Type II fields

- The computations in the case of type II fields, with $v$ the single complex place, are much harder, so I won't talk too much about this.

- The reason is that it's way harder to figure out what $\epsilon$ is supposed to be if only its absolute value is known. In the case where all Archimedean places are real, knowing the absolute value of a quantity determines the quantity up to sign. But in the case of a complex embedding, knowing the absolute value of something is not enough.

- To do things effectively requires additional trickery and using Stark somewhere else, in nicer subfields, to get more information about the Stark units of interest.

- We might hope that the conjecture could be refined further, to say something about the arguments of the Stark units in addition to their absolute values.

## 4 Function Fields

- Jordan's not here, but he would ask, so I'll just say – there is an analogue of Stark's conjectures to the function field case, and it (as often) is easier to prove things there.

### 4.1 Brumer-Stark

- Here's how Brumer-Stark works in the function-field case.

- Let $k$ be a function field in one variable over $\mathbb{F}_q$ and $K/k$ be finite abelian with Galois group $G$. Further assume that the constant field $\mathbb{F}_q$ of $k$ is algebraically closed in $K$.

- Let $S$ be a finite nonempty set of places of $k$ containing all places at which $K$ ramifies.

- The $L$-series becomes $L_S(T, \chi) = \sum_{(\mathfrak{a},S)=1} \chi(\mathfrak{a}) \cdot T^{\deg(\mathfrak{a})} \in E[[T]]$, where $E$ is any characteristic-0 field containing the values of the character $\chi$ of $G$ (considered also as a character on ideals by the reciprocity map), and $\deg(\mathfrak{a})$ is the $\mathbb{F}_q$-dimension of $\mathcal{O}_S/\mathfrak{a}$, where $\mathcal{O}_S$ is the ring of $S$-integers in $K$.

  - For $\chi$ trivial and $T = q^{-s}$ this becomes the usual (congruence) zeta function.
  - By Dwork, the $L$-series is a rational function in $T$.

- The partial zeta functions are $Z_S(T,\sigma) = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a},S)=1 \\ \sigma_a = \sigma}} T^{\deg(\mathfrak{a})} \in \mathbb{Z}[[T]]$.

- By Fourier-inversion we can see that $Z_S(1,\sigma) \in \mathbb{Q}$ and so we can define a Stickelberger element in analogy to the number field case: $\theta_S = \sum_{\sigma \in G} Z_S(1,\sigma) \cdot \sigma^{-1} \in \mathbb{Q}[G]$.
  - Note that $T = 1$ is equivalent to $s = 0$ since $T = q^{-s}$ so we are actually evaluating the zeta functions at the same place as in the number field case.

- Brumer Conjecture for Function Fields (Tate-Deligne): For any divisor $\mathfrak{a}$ of degree 0 in $K$, there exists $\beta$ such that$\mathfrak{a}^{(q-1)\theta_S} = (\beta)$. Furthermore, $K(\beta^{1/(q-1)})$ is an abelian extension of $k$.
  - Tate proved the first statement, and then Deligne proved the second part.
  - Essentially the statement that there exists some ideal class annihilator.
  - This turns out to be essentially the statement that the numerator of the zeta function is the minimal polynomial of Frobenius.
  - But this is part of the Weil conjectures! (Again, Deligne.)
  - There is evidently another proof of the Deligne piece using Drinfeld modules.

### 4.2 Other Function Field Stuff

- The abelian rank-1 Stark conjecture is proved in function fields.
- Quite a few cases for higher rank have also been proved.

## 5 Ending notes

### 5.1 Higher rank, and non-abelian Stark conjectures

- The "principal Stark conjecture" gives information about the leading term of Artin L-functions of (Galois extensions of algebraic) number fields.
  - In general the conjecture states that the leading term should be the "Stark regulator" (which will be the natural log of some algebraic quantity, like the usual regulator) times another algebraic number.
  - But without any information on how complicated this algebraic number could be, it's impossible to attack this numerically.
- Rubin and others generalized to higher rank cases.
- Some stuff is sorta subsumed by the equivariant Tamigawa number conjecture. Maybe. [I won't talk about this.]

### 5.2 The local Stark conjecture

- As one might expect, there is an analogue to the Stark conjecture(s) in the case of local fields.
- I don't want to talk about this, but it exists, and is also interesting. Like the local version of basically everything, it can behave more nicely, and there is hope of a connection between the local and global versions.

### 5.3 AWS

- If you think all of this stuff is cool, you should sign up for the AWS project on the rank-1 abelian Stark conjecture.