What Are Groups, And Why Are They Interesting?

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Northeastern University

Northeastern Math Club

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Outline of Talk

We will start with some general discussion of symmetries of objects in space.

We'll then look at a bunch of examples of symmetries, work out some properties of symmetries, and get some practice composing symmetries.

Then I will present the formalized definition of an abstract group.

Finally, I will mention a few other things about various kinds of groups.

Symmetries, I

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(Pause for audience to be annoyed at having to answer tough philosophical questions this early in the talk.)

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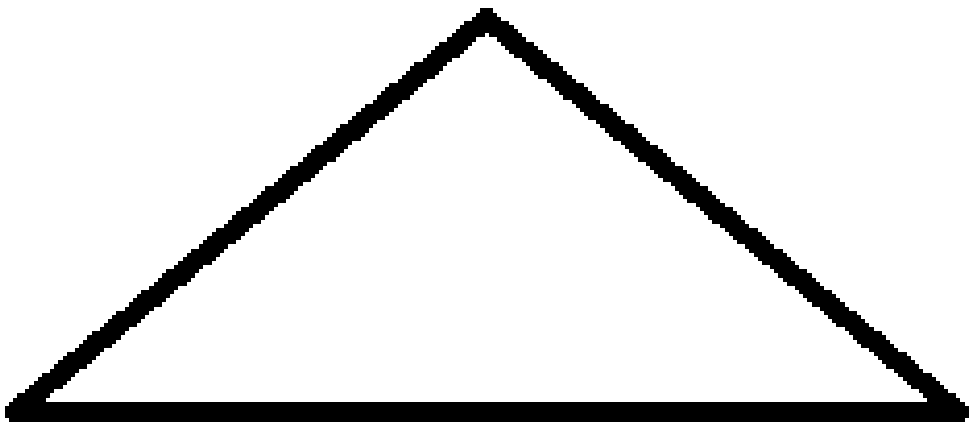
Okay, how about this: I'll give you some pictures and you tell me if they feel symmetric. Ready?

Symmetries, II

Is this symmetric? What symmetries do you see, if any?



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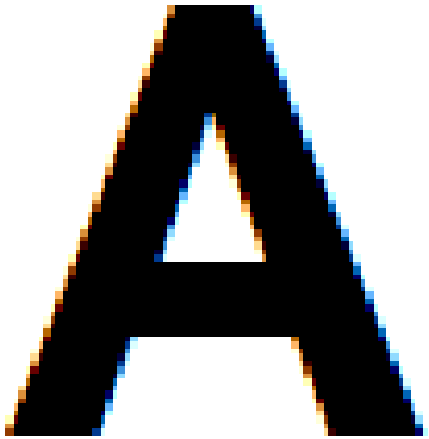

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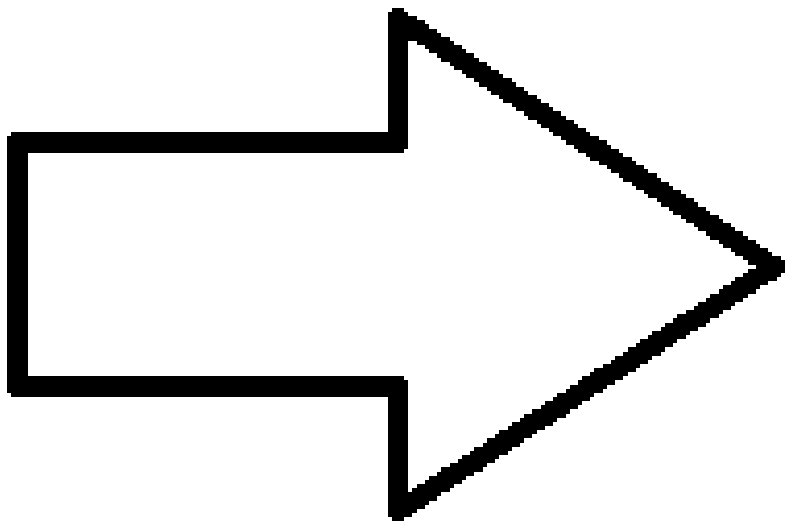

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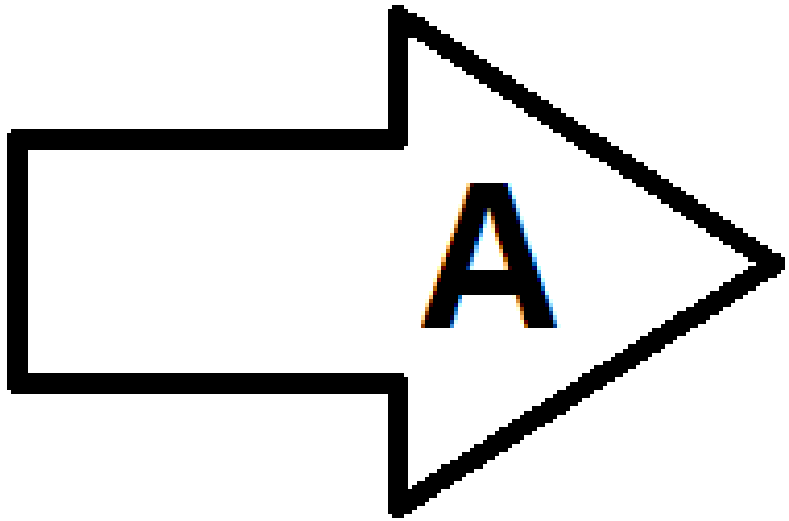

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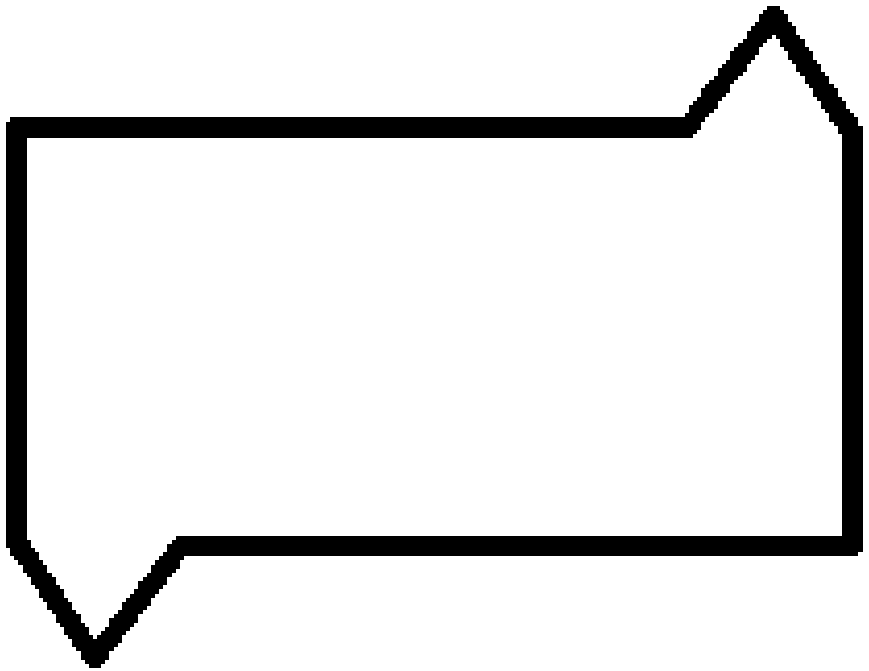

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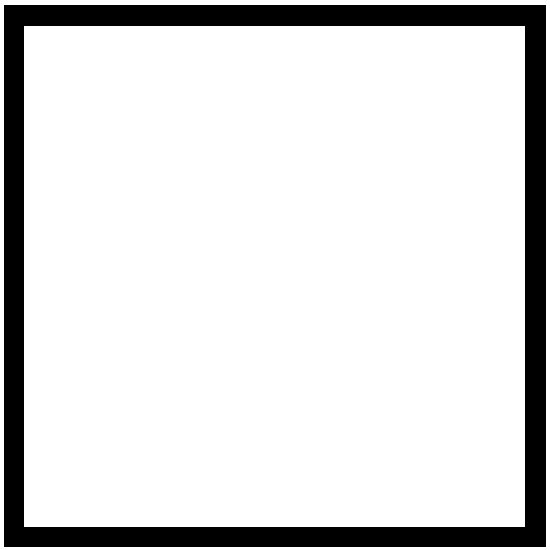

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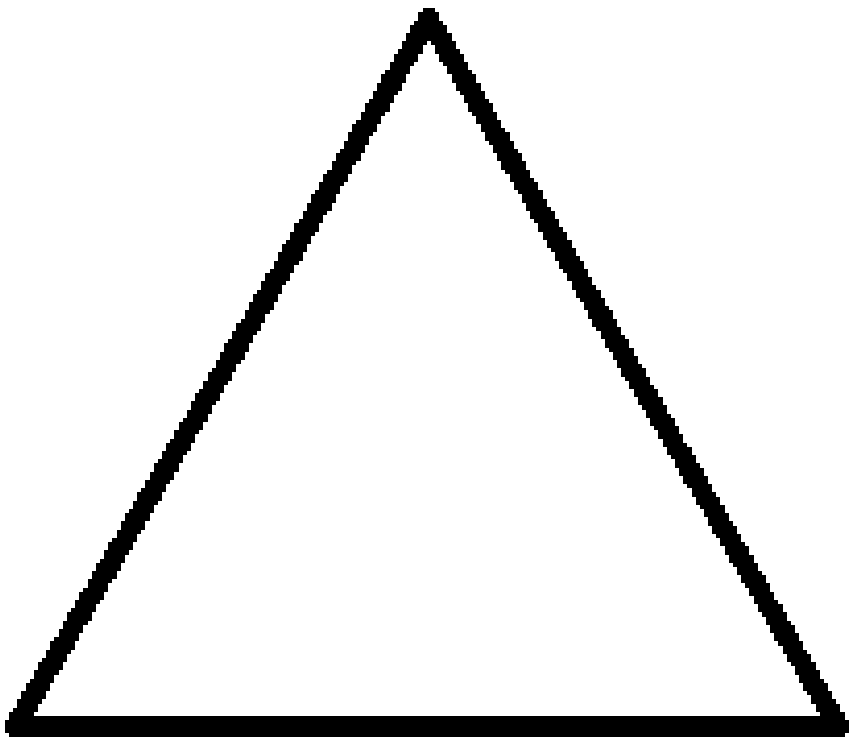

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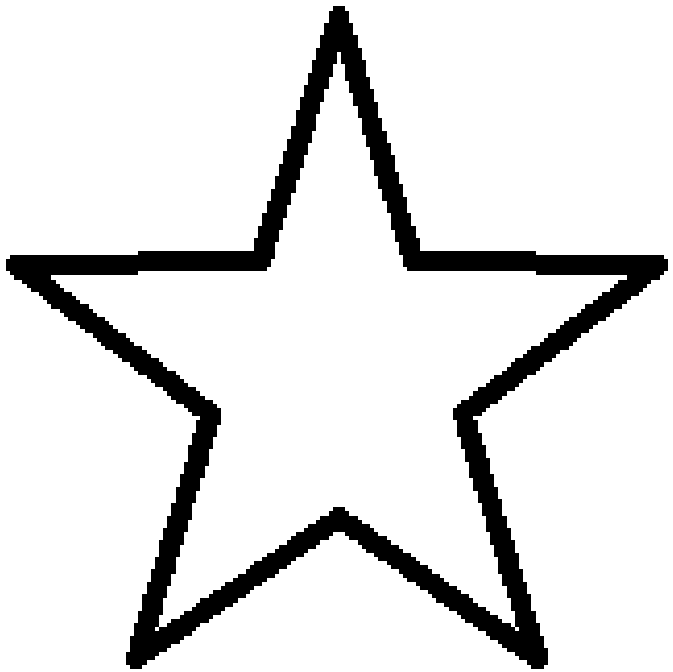

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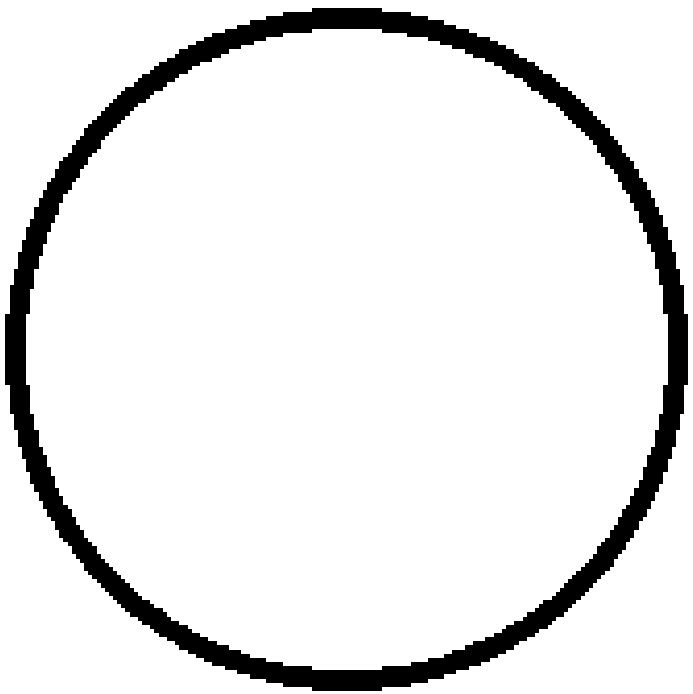
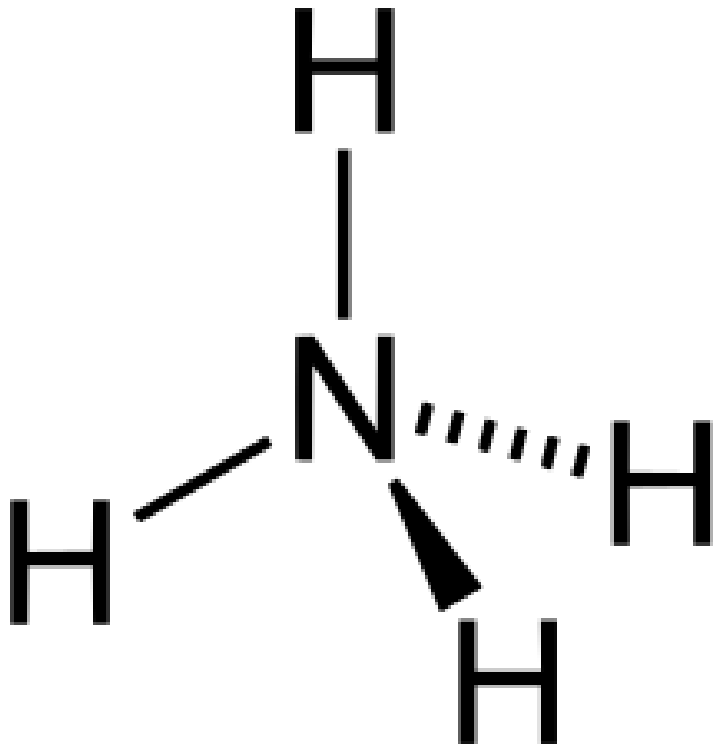


Symmetries, II

Is this symmetric? What symmetries do you see, if any? (The object isn't the literal picture: it's the ammonium molecule NH_4^+, which is shaped like a regular tetrahedron.)



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- Let's try this: an object is "symmetric" when it looks the same after a transformation of some kind, and we call the transformation a "symmetry" of the object.
- We saw some different kinds of symmetries in the examples: some had mirror symmetry (where they look the same after reflecting across a line), while others had rotational symmetry (where they look the same after a rotation).

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Sounds good to me! Let's count some symmetries.

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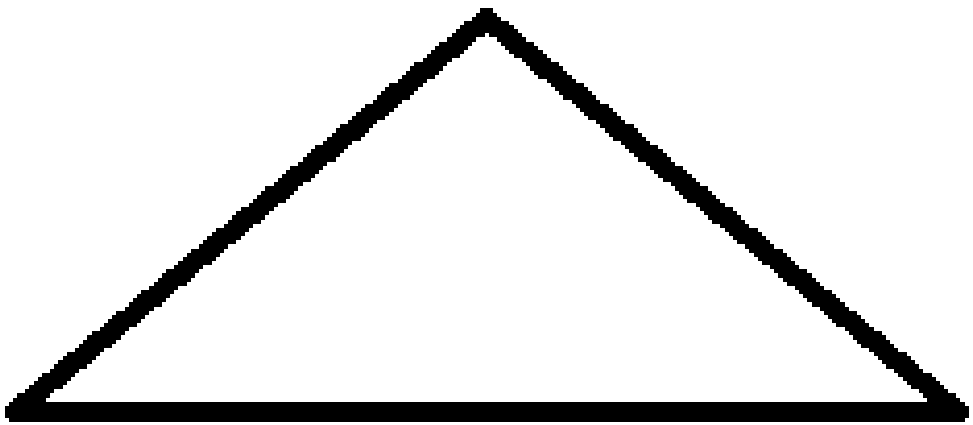

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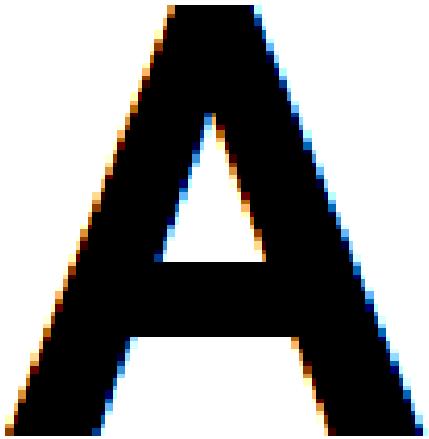

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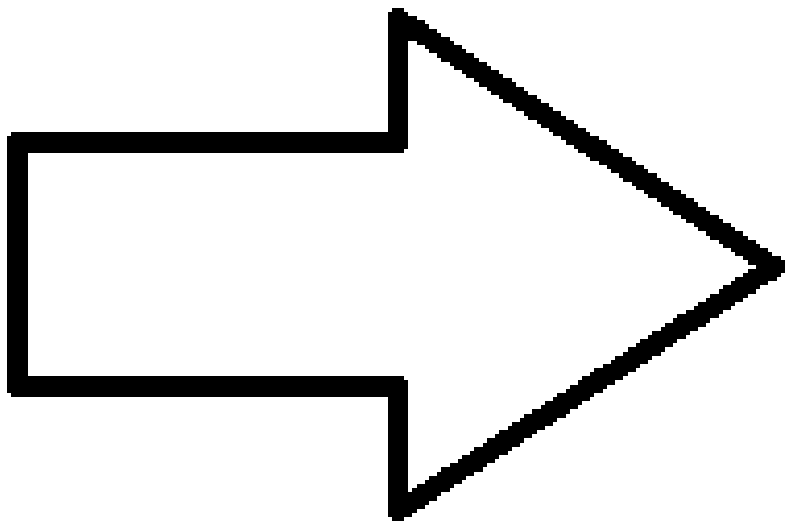

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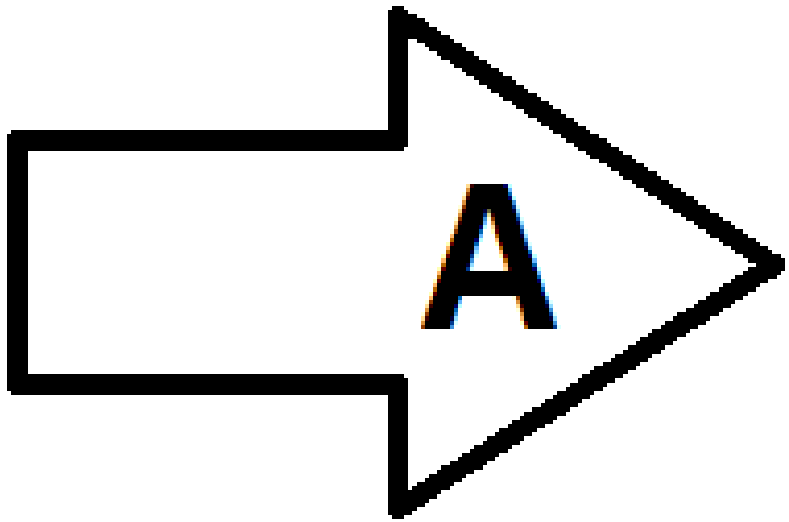

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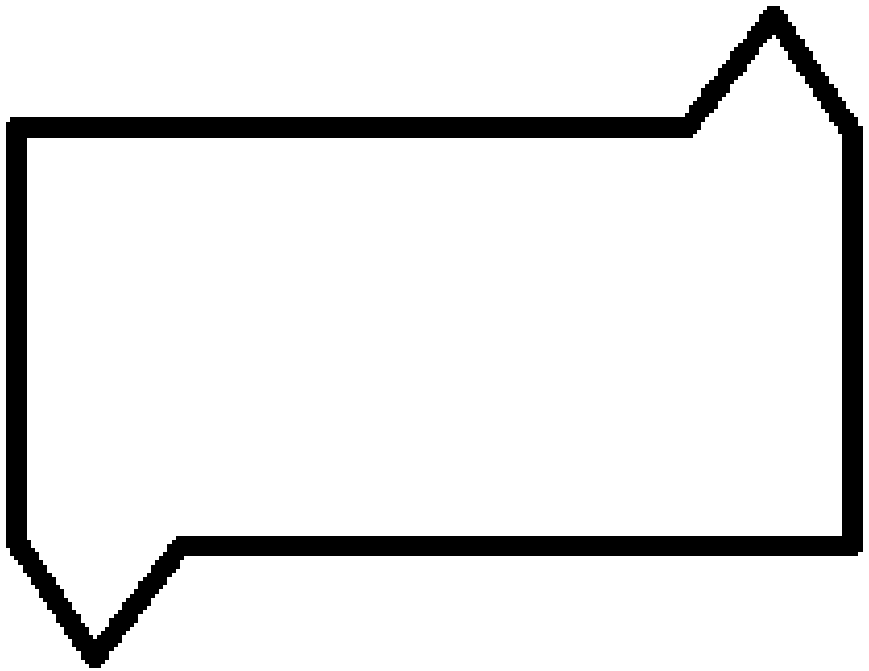

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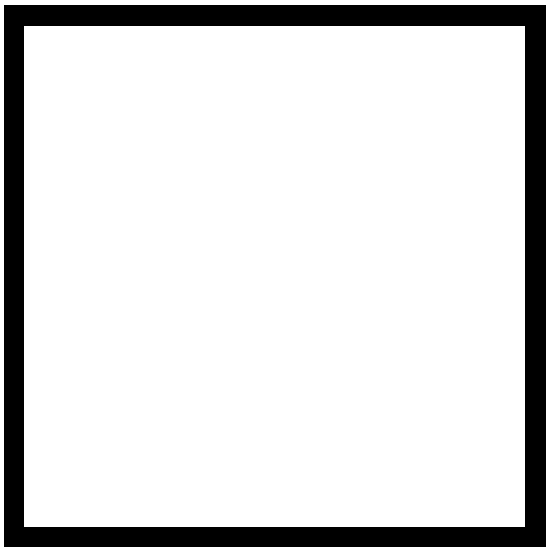

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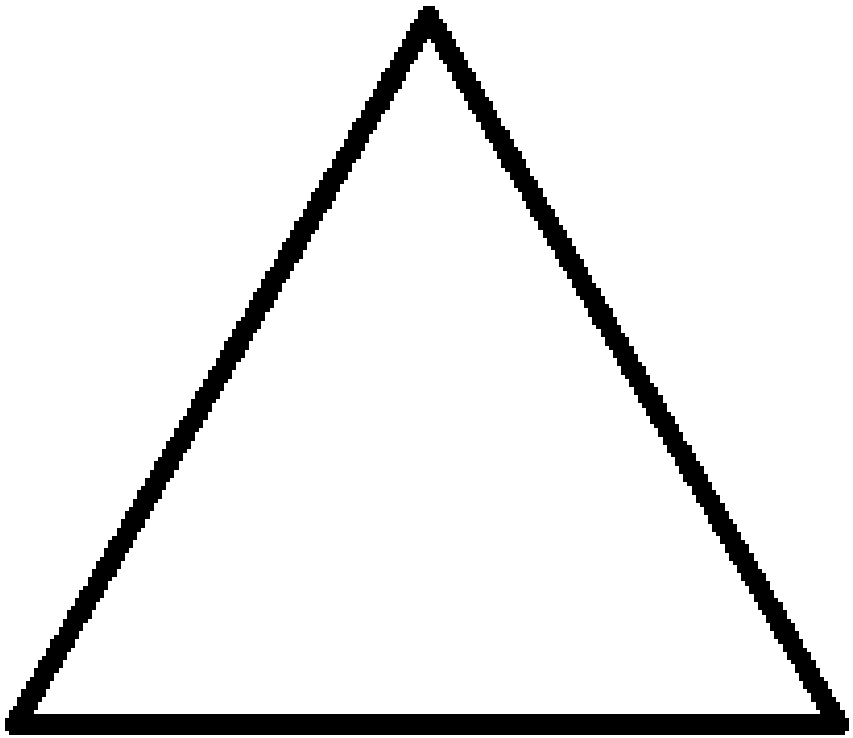

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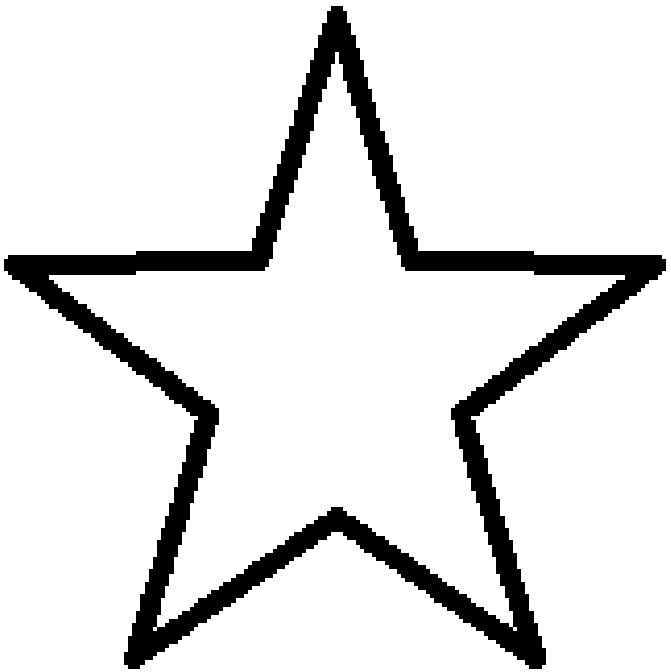

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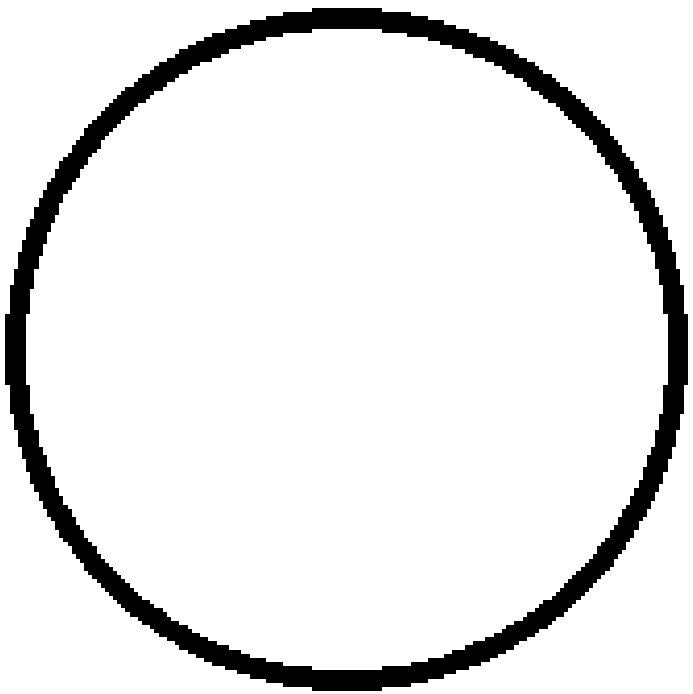

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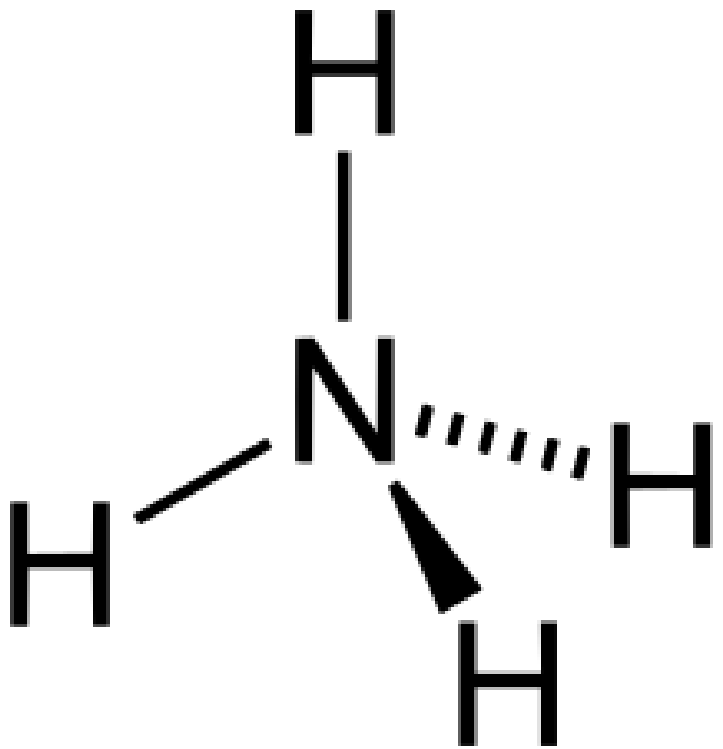
How many symmetries do you count for this object? (Remember this is ammonium, NH_4^+, shaped like a regular tetrahedron.)



Symmetries, V

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1. Every object has at least one symmetry: the identity symmetry, which leaves the object unchanged.
2. Some objects have more symmetries. Sometimes there might just be one more, but other times there can be many.
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Symmetries, VI

Let me highlight one more centrally important fact:

Symmetries are **functions**!

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- Sure it is: after all, as we decided, symmetries are transformations: they transform an object, which is to say, they do something to it.
- Taking an object and doing something to it is exactly what a function does. Therefore, transformations are functions, and so symmetries are functions too.

Symmetries as Functions, I

So now let me ask you this question: what can[1] we do with functions?

[1] Really, what can't we do with functions?

Symmetries as Functions, I

So now let me ask you this question: what can[1] we do with functions?

1. We can give functions names (like f, g, h, ...).
2. We can compose functions.
3. We can ask whether the function has an inverse, and if so we can try to find it.

Great, let's try these!

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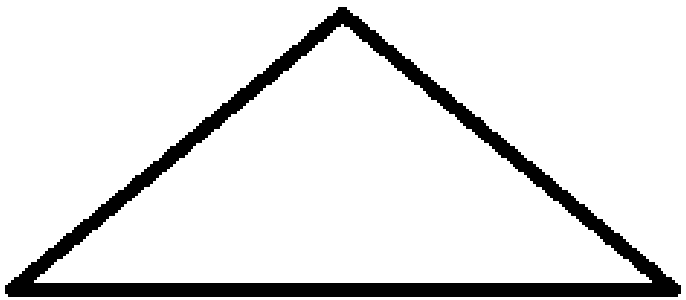
Symmetry Example 1: Isosceles Triangle

So let's think about the two symmetries for this triangle:



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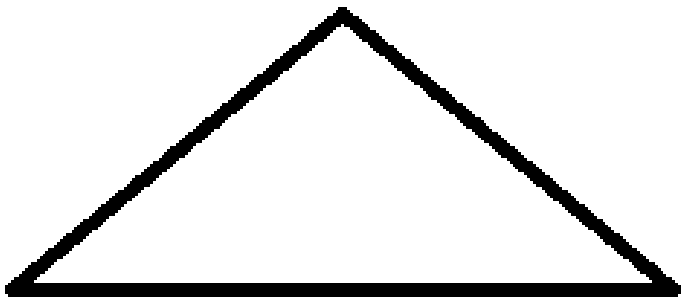

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So: if we pick two symmetries, how do we compose them? Easy enough: just apply the first one, then the second one. Try it now (remember we apply functions right-to-left):

$e \circ e$ $\quad$ $e \circ f$ $\quad$ $f \circ e$ $\quad$ $f \circ f$

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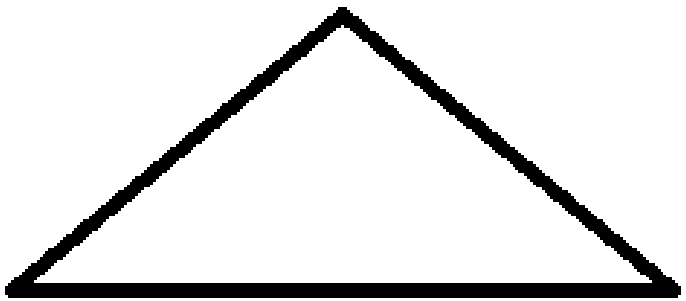

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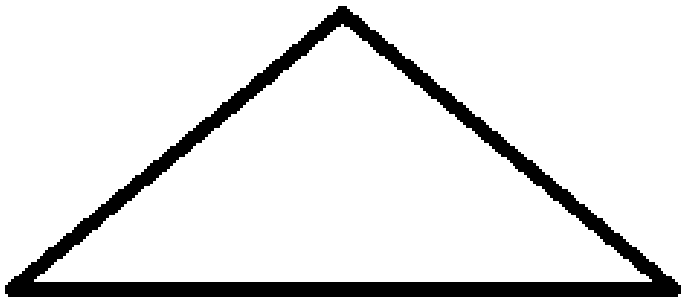

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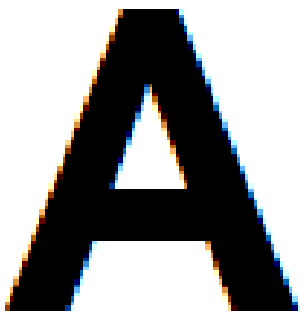
- They both have inverses, and in fact each one is its own inverse.
- We can see that just by noting that if we do a symmetry twice, we get back where we started.

Symmetry Example 2: A, eh?

How about the two symmetries for the letter A?



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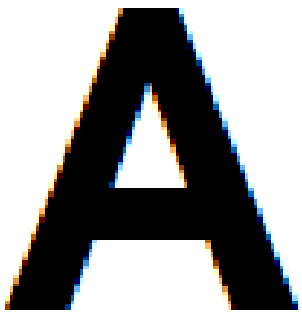

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Hmmm... this is kind of suspicious: that description sounds really similar to the symmetries of the triangle we just talked about.[2] Do the compositions and inverses work the same way as before?

(Pause to let the audience come to that conclusion.)

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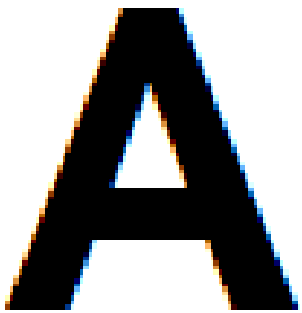

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Yep, it's exactly the same.

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Symmetry Example 3: Rectangle

How about the four symmetries for the rectangle?

- The identity e.
- The vertical reflection v.
- The horizontal reflection h.
- The π-radian rotation r.

In total we're going to have 16 possible compositions to calculate here. Here's the thing: it can be hard to keep track of what happened when you do two of them in a row.

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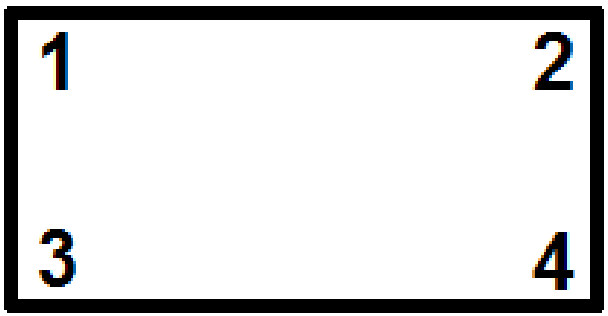
Here's a great way to do it: actually do these symmetries on a real rectangle! Luckily enough, I brought some for you, and I even labeled the vertices so you can keep everything straight.

Symmetry Example 3: Rectangle

Let me label the rectangle for you:



Our symmetries:

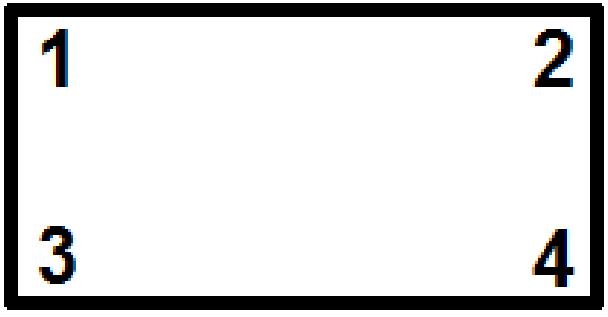
- The identity e.
- The vertical reflection v.
- The horizontal reflection h.
- The π-radian rotation r.

Now, your task: figure out what all 16 compositions are!
(Remember, functions apply right-to-left.)

$e \circ e$	$e \circ v$	$e \circ h$	$e \circ r$
$v \circ e$	$v \circ v$	$v \circ h$	$v \circ r$
$h \circ e$	$h \circ v$	$h \circ h$	$h \circ r$
$r \circ e$	$r \circ v$	$r \circ h$	$r \circ r$

Which symmetries have inverses? What are the inverses?

Symmetry Example 3: Rectangle



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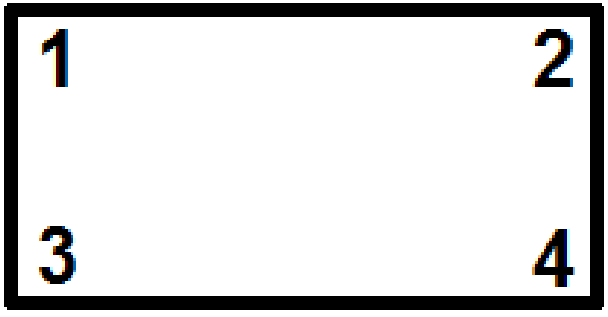

Our symmetries:

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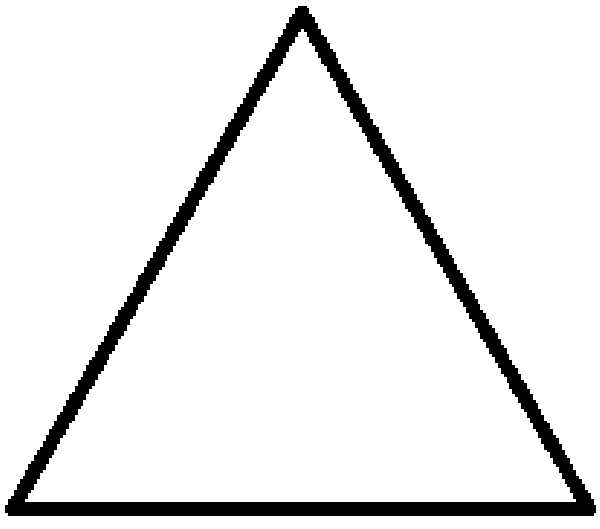
Let's summarize the results:

- Composing any two symmetries gives another symmetry.
- Composing e with any symmetry gives that symmetry back.
- Composing any of v, h, or r with itself gives the identity.
- Composing two of v, h, r in either order gives the other one.
- Every symmetry has an inverse, and in fact every symmetry is its own inverse.

Symmetry Example 4: Equilateral Triangle

Let's now take a look at the symmetries of an equilateral triangle. Here, things get a bit more exciting.



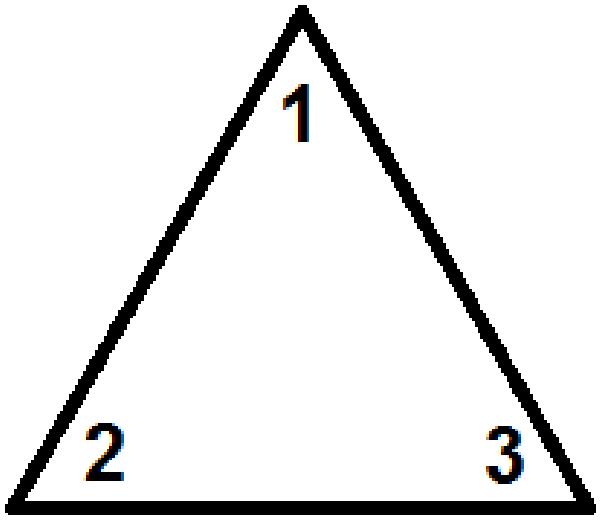
There are six of these symmetries:

- The identity e.
- The two rotations (1/3 turn and 2/3 turn, counterclockwise).
- The three reflections (across the three altitudes).

Now, we could give each of these symmetries their own label. But then we'd have to remember what they all mean, and that will get annoying. Let's try to be more economical and try to write some symmetries in terms of others.

Symmetry Example 4: Equilateral Triangle

I claim that we can write all the other symmetries in terms of just two: the 1/3-counterclockwise turn r and the vertical reflection s. Let's label the triangle and have you move it around yourself, like with the rectangle:



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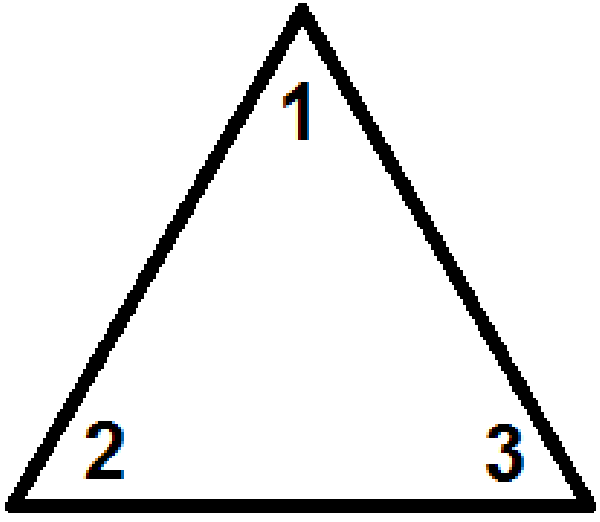

So now, convince yourself that our six symmetries are these:

- The identity e.
- The 1/3-CCW turn r.
- The 2/3-CCW turn $r \circ r$.
- The vertical reflection s.
- Another reflection $s \circ r$.
- The last reflection $s \circ r \circ r$.

Notational Interlude

I personally am getting tired of all those function composition symbols $\circ$. Can we agree that we don't really need to write them anymore? (Pause to intimidate audience into agreeing.)

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- So, we've "agreed" that we can just write sr instead of $s \circ r$.
- We shouldn't have any danger of confusion, as long as we remember that these letters still represent functions and so we apply them from right to left.

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I personally am getting tired of all those function composition symbols $\circ$. Can we agree that we don't really need to write them anymore? (Pause to intimidate audience into agreeing.)

- So, we've "agreed" that we can just write sr instead of $s \circ r$.
- We shouldn't have any danger of confusion, as long as we remember that these letters still represent functions and so we apply them from right to left.

Also, it looks weird to write something like rr, don't you think? Can we just write r^2 for that, and agree that r^2 just means r applied twice?

Notational Interlude, II

Okay great! One more thing though: if we're going to write everything like it's multiplication, do we have to be careful about parentheses? In other words, is there a difference between, say, $(sr)e$ and $s(re)$?

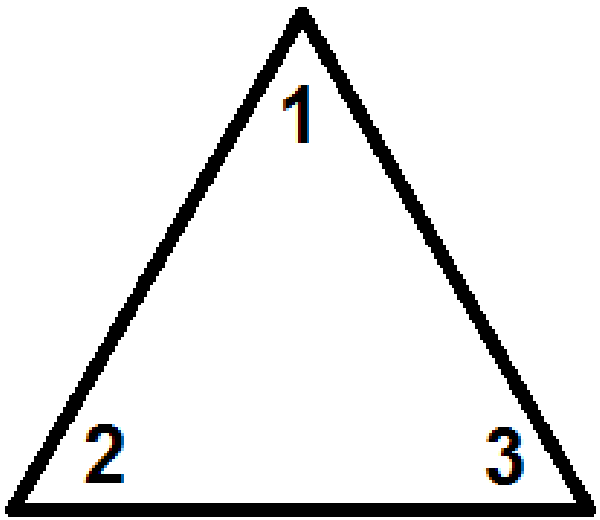
Notational Interlude, II

Okay great! One more thing though: if we're going to write everything like it's multiplication, do we have to be careful about parentheses? In other words, is there a difference between, say, $(sr)e$ and $s(re)$?

- Answer: no! Why not? Because these are functions!
- Think of these symmetries r, s, and e as a function of an actual variable x for a moment.
- What does $(sr)e$ do to x? It computes $[(sr)e](x) = [sr](e(x)) = s(r(e(x)))$.
- What does $s(re)$ do to x? It computes $[s(re)](x) = s(re(x)) = s(r(e(x)))$.
- Those are exactly the same thing! They just say what happens to x after we apply e first, r second, and s third.

Symmetry Example 4: Equilateral Triangle

So let's write down our symmetries using this simplified notation:



Here they are:

- The identity e.
- The 1/3-CCW turn r.
- The 2/3-CCW turn r^2.
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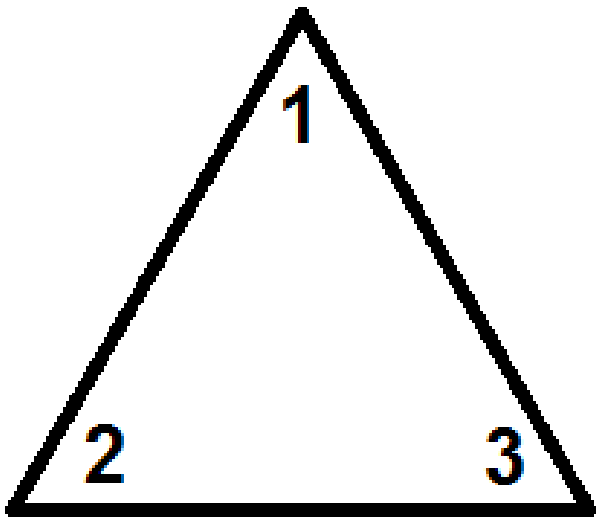

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Okay now, how about you try out a few compositions? I'll leave some multiplication dots so you can tell what we're composing.

1. $r \cdot r \cdot r$
2. $r \cdot r^2$
3. $r^2 \cdot r$
4. $r^2 \cdot r^2$
5. $s \cdot s$
6. $r \cdot s$
7. $sr \cdot sr$
8. $s \cdot r \cdot s \cdot r$
9. $s \cdot rs \cdot r$
10. $sr^2 \cdot sr^2$
11. $s \cdot sr$
12. $sr \cdot s$

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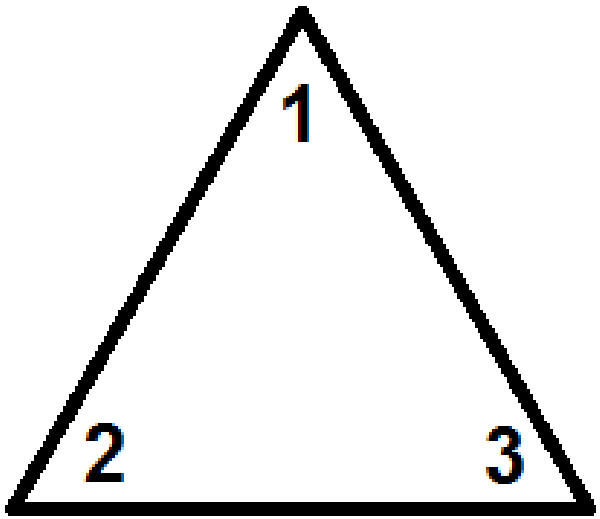

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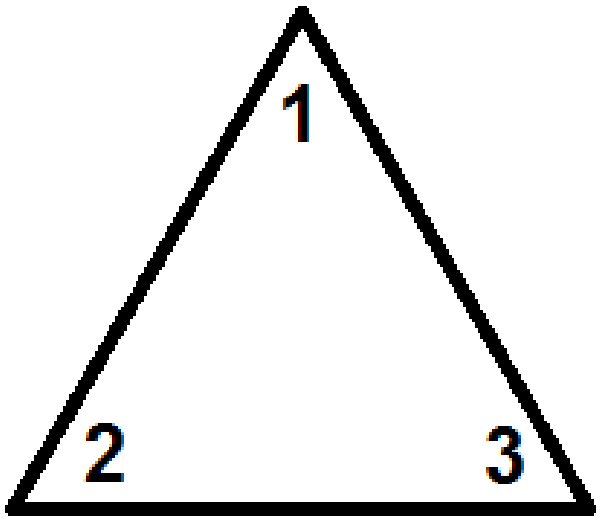

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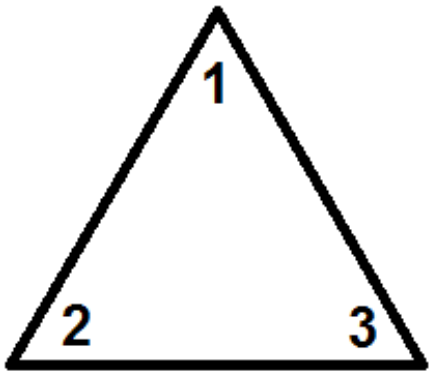
Which symmetries have inverses? What are they?

1. $e^{-1} = e$
2. $r^{-1} = r^2$
3. $(r^2)^{-1} = r$
4. $s^{-1} = s$
5. $(sr)^{-1} = sr$
6. $(sr^2)^{-1} = sr^2$.

Symmetry Example 4: Equilateral Triangle



Some interesting results:

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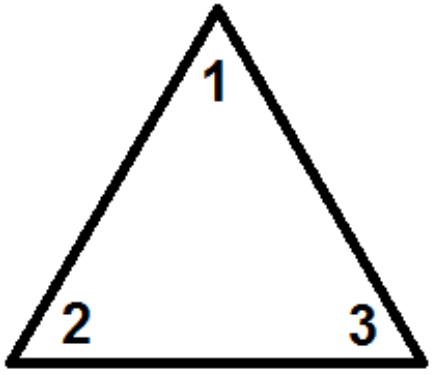
Symmetry Example 4: Equilateral Triangle



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Some interesting results:

- Composing any two symmetries gives another symmetry.
- Composing e with any symmetry gives that symmetry back.
- Every symmetry has an inverse.
- Composition of symmetries is associative: $(ab)c = a(bc)$.
- We have $rs = sr^2$. In particular notice that rs does NOT equal sr! (Is that sensible? Sure: functions don't usually have the property that $f \circ g = g \circ f$, so why would symmetries?)

Common Themes

Let's now write down the common themes we saw with symmetries.

0. Composing any two symmetries gives another symmetry.
1. Composition of symmetries is associative: $(ab)c = a(bc)$ for any symmetries a, b, c.
2. There is an identity symmetry e such that $ea = a = ae$ for any other symmetry a.
3. Every symmetry a has an inverse symmetry a^{-1} such that $aa^{-1} = e = a^{-1}a$.

Any objections? (Don't pause; ignore any objections.)

Groups

Okay, so now let me tell you about something completely unrelated: the definition of a group.

Definition (Groups)

A group is a set G together with a composition operation $\circ$ satisfying the following properties:

0. *Composing any two elements of G gives another element of G.*
1. *Composition is associative: $(g \circ h) \circ k = g \circ (h \circ k)$ for any g, h, k in G.*
2. *There is an identity element e such that $e \circ g = g = g \circ e$ for any g in G.*
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Huh. This seems familiar for some reason.

Groups, II

In fact, the whole idea of a group is just a way of trying to understand symmetries!

- We just abstracted the simple observations we made about symmetries into a general definition: nothing more.
- All we need is an operation on a set that is associative, has an identity element, and has inverses. That's all a group is!
- Totally not scary at all, right? Well...

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- Totally not scary at all, right? Well... part of the reason groups have a bit of a fearsome reputation is because they have so few properties, it's hard to establish general properties that apply to all groups.

But now that we have an abstract definition, let me try to convince you that have heard about a few other groups already.

Groups, III

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- Does every real number have an additive inverse? Yep, just as before for any a just use its negative $-a$: $a + (-a) = 0 = (-a) + a$, so ✓

Groups, V

The last two examples were what we often call "additive groups": groups of numbers under addition. Some more examples (in decreasing order of how likely you are to know what they mean):

- The rational numbers under addition.
- The complex numbers under addition.
- Vectors in 2-dimensional space under addition.
- 2×2 matrices under addition.
- The integers modulo n under addition.
- The quaternions under addition.
- The points on an elliptic curve under geometric addition.
- The hyperreal numbers under addition.

We also get similar sorts of groups arising from multiplication, which naturally are called "multiplicative groups".

Groups, VI

Question: Is the set of real numbers with operation $\cdot$ a group?

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- Is there a multiplicative identity? Yep, since famously $1 \cdot a = a = a \cdot 1$ for any a, so ✓
- Does every real number have a multiplicative inverse?

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- Let's check: if we multiply two real numbers, do we get another real number? ✓
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- Is there a multiplicative identity? Yep, since famously $1 \cdot a = a = a \cdot 1$ for any a, so ✓
- Does every real number have a multiplicative inverse? Hmmm... is it true that for any real number a, we have some other real number b with $ab = 1$? Well, we'd need $b = 1/a$ but... what if $a = 0$? We can't divide by zero! So NO!

We can see here that 0 causes trouble. What if we get rid of it?

Groups, VII

Question: Is the set of nonzero real numbers with operation $\cdot$ a group?

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- Does every nonzero real number have a multiplicative inverse?

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- Is there a multiplicative identity? Yep, since famously $1 \cdot a = a = a \cdot 1$ for any a, so ✓
- Does every nonzero real number have a multiplicative inverse? Yes, since now we can use $1/a$ as the inverse of a, so ✓.

Groups, VIII

There are lots of other multiplicative groups like the one we just discussed. Here are some.

- The set $\{1, -1\}$ under multiplication. (Try it!)
- The nonzero rational numbers under multiplication.
- The nonzero complex numbers under multiplication.
- The set $\{1, i, -1, -i\}$ under multiplication (where $i = \sqrt{-1}$).
- The invertible 2×2 matrices (i.e., ones with nonzero determinant) under multiplication.
- The nonzero quaternions under multiplication.

Symmetry Groups, I

But the real motivation for groups is to consider the symmetry group of an object.

- Broadly speaking, here's how that works.
- If X is an object, its "symmetry group" is the set of all symmetries of X: these are the invertible transformations (functions) from X to X that preserve the structure of X.

Why does the collection of symmetries form a group?

- The composition of two symmetries is also a symmetry (it's also a function, and it preserves X).
- The composition of symmetries is associative, since they're functions.
- There's an identity symmetry: the identity function, which leaves X alone.
- Finally, every symmetry has an inverse by definition.

Symmetry Groups, II

And in fact, we already wrote down some symmetry groups earlier.

- The symmetry group of the rectangle had four elements: the identity e, the horizontal reflection h, the vertical reflection v, and the 1/2-turn rotation r.

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- The symmetry group of the rectangle had four elements: the identity e, the horizontal reflection h, the vertical reflection v, and the 1/2-turn rotation r.
- The symmetry group of the equilateral triangle had six elements: with one rotation r and one reflection s, the elements were $\{e, r, r^2, s, sr, sr^2\}$.

A classical problem in group theory was to write down the symmetry groups of other possible shapes. (In fact, this was also of substantial interest to chemists: you can imagine why if you remember the ammonium example from earlier!)

Symmetry Groups, III

Just in case you want to know the symmetry groups for some other shapes, here are some:

- The symmetry group of the regular n-gon is called the dihedral group of order $2n$. It has $2n$ elements that can be written in terms of a rotation r and a reflection s: they are $\{e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$. (It's very similar to the equilateral triangle, which is the $n = 3$ case.)

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- The symmetry group of the regular tetrahedron (not allowing reflections) is a group called A_4, which has 12 elements. Allowing reflections, the group is bigger: it's called S_4 and has 24 elements.
- The symmetry group of the circle (not allowing reflections) is a group called S^1, and (for the initiated) it's isomorphic to the quotient $\mathbb{R}/\mathbb{Z}$. If reflections are allowed, the group is bigger: it's the semidirect product $S^1 \rtimes (\mathbb{Z}/2\mathbb{Z})$.

Symmetry Groups, IV

Another wonderful thing we can do is consider the symmetries of mathematical objects, rather than geometric ones.

- Remember: a symmetry is just a structure-preserving function on an object X.
- Here X can be any kind of object, not necessarily a geometric figure.

Symmetry Groups, V

Question: What are the "symmetries" of the set $X = \{1, 2, 3\}$?

- The "structure-preserving functions" here would just be functions from X to X that have inverses. In other words: they are the permutations of the set X.
- There are 6 of them in total, and we could just list them all in terms of how they rearrange the elements $1, 2, 3$.
- The identity function would send $1 \to 1$, $2 \to 2$, $3 \to 3$.
- Another function sends $1 \to 2$, $2 \to 3$, $3 \to 1$. (And so on.)

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- Another function sends $1 \to 2$, $2 \to 3$, $3 \to 1$. (And so on.)

Groups of this kind are called "permutation groups". The symmetric group on n objects, denoted S_n, is the permutation group of the set $\{1, 2, \ldots, n\}$.

Wrap-Up

There is a vast amount more to say about groups, of course.

- We have an entire semester-long course about groups: Math 3175, which all of the math majors are required to take. But this course barely scratches the surface of all the things one can learn about groups and do with groups.

Wrap-Up

There is a vast amount more to say about groups, of course.

- We have an entire semester-long course about groups: Math 3175, which all of the math majors are required to take. But this course barely scratches the surface of all the things one can learn about groups and do with groups.
- Groups show up in almost every area of mathematics, particularly in algebra, number theory, geometry, topology, and combinatorics, but certainly also in analysis, applied mathematics, and even logic.
- Outside of mathematics, groups are also quite important in physics, chemistry, and computer science and even sometimes show up in such disparate fields like art (in the study of artistic patterns) and anthropology (in the study of certain kinship arrangements).

Thanks!

Thanks to Zack Eisbach and the other math club organizers for providing me the opportunity to speak here today!

Please also allow me to advertise the Putnam Club, which meets Wednesdays from 6pm-7:30pm in 509 Lake. We get together to (try to) solve some problems from old Putnam exams, and also eat pizza. If you like competition math and/or problem-solving, come check us out!

I hope you enjoyed my talk, and I'd like to thank you for attending! Enjoy your weekend!