The Kakeya Problem in Algebra and Analysis

Evan Dummit

Northeastern University Math Club

February 27, 2020

Outline of Talk

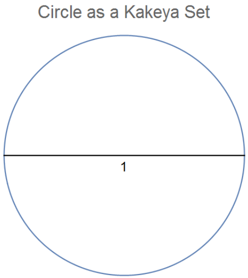
1. The Kakeya problem in analysis
2. The Kakeya problem over finite fields
3. The Kakeya problem over local rings
4. Open questions

The Kakeya Needle Problem, I

Definition (S. Kakeya, 1917)

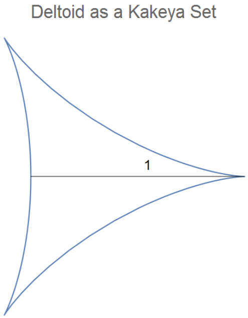
A **Kakeya needle set** *is a subset of the plane inside which it is possible to rotate a needle of length 1 completely around.*

An example: a circle of diameter 1 (area $\pi/4$):



The Kakeya Needle Problem, II

Another example: a deltoid (area $\pi/8$):



The Kakeya Needle Problem, III

Question

What is the minimum area of a Kakeya needle set?

It was originally believed that the deltoid example (of area $\pi/8$) was the smallest possible Kakeya set. But....

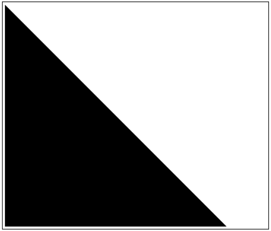
Theorem (A. Besicovitch, 1919)

There exists a Kakeya needle set in the plane having arbitrarily small area.

The Kakeya Needle Problem, IV

Basic idea for constructing a Kakeya set of small area:

- Start with a simple Kakeya set.
- Slice up the set into pieces.
- Slide the the pieces together so that they overlap a lot.
- Repeat steps 2-3 until the set is arbitrarily small.



The Kakeya Needle Problem, V

What about higher dimensions? The key idea is that we can orient the needle in any direction:

Definition

For $n \geq 2$, a **Kakeya set** *is a set in $\mathbb{R}^n$ inside which it is possible to rotate a needle of length 1 to point in any direction.*

If K is a Kakeya set in the plane of small measure, then $K \times [0, 1]^{n-2}$ is a Kakeya set of equal measure in $\mathbb{R}^n$.

Continuous motion implies there is no Kakeya set of measure zero (not so trivial to prove as it might seem!).

Besicovitch and Kakeya Sets

Also of interest is a modified version of the problem with a weaker hypothesis:

Definition

A **Besicovitch set** *is a set of points in Euclidean space which contains a unit line segment in every direction.*

Any Kakeya set is certainly a Besicovitch set, but we can have Besicovitch sets of area zero! (Take an appropriate limit in the construction described earlier.)

Dimension, I

Besicovitch sets can be very small in measure. But there are other notions of size!

Definition

The **Minkowski dimension** *of a set K is defined to be*

$$\dim(K) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

where $N(\epsilon)$ is the number of boxes of side ϵ needed to cover K.

Motivation: if we cover the set with ϵ-boxes, how fast does this number grow in terms of ϵ? For a line, ϵ^{-1}; for a square, ϵ^{-2}, for a cube, ϵ^{-3}, and so forth.

Dimension, II

Other flavors of dimension exist also (e.g., Hausdorff dimension) but they are often harder to use. So what can we say about the Minkowski dimension of a Besicovitch set?

Theorem (R. Davies, 1971)

Any Besicovitch (or Kakeya) set in $\mathbb{R}^2$ has Minkowski dimension 2.

What about in higher dimensions?

Conjecture (Kakeya Conjecture)

Any Besicovitch (or Kakeya) set in $\mathbb{R}^n$ is of Minkowski dimension n.

Unfortunately, we only have lower bounds when $n > 2$. There are various trivial bounds (on the order of things like $\sqrt{n}$ or $n/2$).

Dimension, III

At this point, you might wonder: who is interested in this problem?

Dimension, III

At this point, you might wonder: who is interested in this problem?

Pictured: Terence Tao, IMO gold medalist (age 13), Princeton PhD (age 21), UCLA Professor (age 24), Fields Medalist (age 31), coauthor of over 300 papers and 17 books.

Dimension, IV

More substantial:

Theorem (T. Wolff, 1995)

Any Besicovitch (or Kakeya) set in $\mathbb{R}^n$ has Minkowski dimension at least $(n+2)/2$.

This was improved for $n > 4$:

Theorem (N.H. Katz, T. Tao, 1995)

Any Besicovitch (or Kakeya) set in $\mathbb{R}^n$ has Minkowski dimension at least $(1/\alpha)n + (1-\alpha)/\alpha \approx 0.596n + 0.403$, where $\alpha^3 - 4\alpha + 2 = 0$.

The proofs of these theorems are very hard.

Kakeya Sets in Finite Fields, I

Let's now look at the Kakeya problem in a finite field. Definitions:

- Let $\mathbb{F}_q$ be a finite field, and n a fixed positive integer.
- Space of interest: $S = \mathbb{F}_q^n$.
- Lines in S are of the form $\{x + sy \; : \; s \in \mathbb{F}_q, \, x, y \in S, y \neq 0\}$.
- A direction in S is a class of y giving the same line.

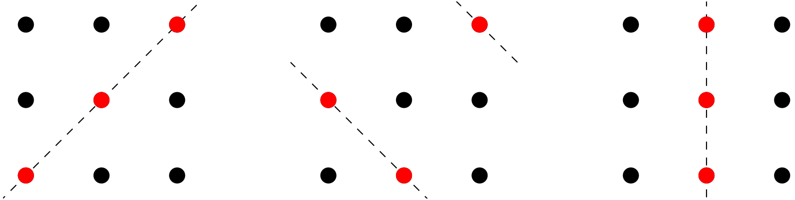
Definition

A **Kakeya set** *is a set of points in $\mathbb{F}_q^n$ which contains a line in every direction.*

By "contains a line" we mean "contains the q points on the line". We dispense with the "length 1" part because everything is finite.

Kakeya Sets in Finite Fields, II

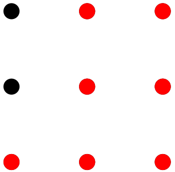
For example, if $n = 2$ and $q = p$ is prime, then we are simply looking at a $p \times p$ grid of points, where lines "wrap around".

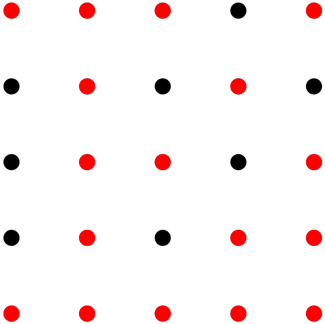


Each line contains p points and there are $p + 1$ possible directions.

Kakeya Sets in Finite Fields, III

Here are some examples of Kakeya sets, in $\mathbb{F}_3^2$ and $\mathbb{F}_5^2$:





Sizes of Kakeya Sets, I

So how small can a Kakeya set in $\mathbb{F}_q^n$ be?

Proposition

Any Kakeya set in $\mathbb{F}_q^2$ contains at least $\frac{1}{2}q^2$ points.

Sizes of Kakeya Sets, II

So how small can a Kakeya set in $\mathbb{F}_q^n$ be?

Proposition

Any Kakeya set in $\mathbb{F}_q^2$ contains at least $\frac{1}{2}q^2$ points.

Proof: The first line has q points, the second adds at least $q-1$ new points, the third adds at least $q-2$ more, ... , yielding at least $\frac{q(q+1)}{2} > \frac{1}{2}q^2$ points in total. □

Reframing: a Kakeya set in $\mathbb{F}_q^2$ contains a positive proportion of the points in $\mathbb{F}_q^2$, and has Minkowski dimension 2.

Sizes of Kakeya Sets, III

Conjecture (Finite-Field Kakeya Conjecture)

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $c_n q^n$ points, for some constant $c_n > 0$.

Originally posed by Wolff in 1999. This problem seemed as hard as Kakeya in $\mathbb{R}^n$:

Theorem (G. Mockenhaupt, T. Tao, 2004)

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $c_n q^{(4n+3)/7}$ points, for a constant $c_n > 0$.

Their proof is quite intricate and analytically-flavored, and any substantial improvement would seem to require very different ideas.

Sizes of Kakeya Sets, IV: A New Hope

Theorem (Z. Dvir, 2008)

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $\binom{n+q-1}{n} \geq \frac{q^n}{n!}$ points.

In other words, a Kakeya set in $\mathbb{F}_q^n$ always has Minkowski dimension n, and contains a positive proportion of the points in $\mathbb{F}_q^n$ as $q \to \infty$. (Thus, the Kakeya conjecture over $\mathbb{F}_q$ is true.)

Dvir's Proof of the Finite-Field Kakeya Conjecture

Dvir's proof is very simple: suppose K has $< \binom{n+q-1}{n}$ points.

- By nullity-rank, there is a nonzero polynomial P in $\mathbb{F}_q[x_1, \ldots, x_n]$ of degree at most $q-1$ vanishing on K.
- Let $P = P_0 + P_1 + \cdots + P_{q-1}$ where P_i is homog. of degree i.
- Because P vanishes on a line in the direction y, there exists b such that $P(b+ty) = 0$ for all t in $\mathbb{F}_q$.
- Then $P(b+ty)$ is a polynomial of degree at most $q-1$ in t having q roots in $\mathbb{F}_q$, so it is the zero polynomial.
- Coefficient of t^{q-1} in $P(b+ty)$ is $P_{q-1}(y)$.
- But then $P_{q-1}(y) = 0$ for all y in $\mathbb{F}_q^n$, meaning that $P_{q-1} = 0$.
- Repeat for the other terms, to conclude P is zero. Contradiction.

The Polynomial Method

Dvir's proof is a stunning example of the "polynomial method": consider a polynomial vanishing on the set, and then prove something about it. Other applications of the polynomial method:

- Sizes of cap sets (sets avoiding 3-term arithmetic progressions, made famous in the card game "Set").
- Erdős distinct distances problem: given n points in the plane, what is the smallest number of distinct distances between the points in terms of n? (Answer: $\geq cn/\log(n)$ for some $c > 0$.)
- Finite-field Nikodym problem, joints problem, and other variations on point-line configurations.

Sizes of Kakeya Sets, V

Distressing caveat: Dvir's proof gives no real information about what Kakeya sets actually look like!

Some improvement in the bound is available, using a slightly more complicated version of the technique:

Theorem (Z. Dvir, S. Kopparty, S. Saraf, M. Sudan; 2009)

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $(\frac{1}{2} + o(1))^n q^n$ points.

The constant is believed to be essentially sharp, up to possible refinement of the $o(1)$.

Between $\mathbb{R}$ and $\mathbb{F}_q$

In $\mathbb{R}^n$ there exist Kakeya sets of measure zero, but over $\mathbb{F}_q^n$, there exists a hard lower bound on measure (independent of q). So perhaps $\mathbb{F}_q$ is not the best analogy for $\mathbb{R}$.

- One possible reason: $\mathbb{F}_q$ has no notion of "distance".
- Points in $\mathbb{F}_q$ are either the same or they're not, unlike $\mathbb{R}$ which has many different distances.

Between $\mathbb{R}$ and $\mathbb{F}_q$

In $\mathbb{R}^n$ there exist Kakeya sets of measure zero, but over $\mathbb{F}_q^n$, there exists a hard lower bound on measure (independent of q). So perhaps $\mathbb{F}_q$ is not the best analogy for $\mathbb{R}$.

- One possible reason: $\mathbb{F}_q$ has no notion of “distance”.
- Points in $\mathbb{F}_q$ are either the same or they’re not, unlike $\mathbb{R}$ which has many different distances.
- Also, notice that as $n \to \infty$, the constant $(\frac{1}{2} + o(1))^n$, representing the density of a Kakeya set in $\mathbb{F}_q^n$, goes to zero.
- Perhaps this may be because there is a Kakeya set in some limit space that “looks like” $\lim_{n\to\infty} \mathbb{F}_q^n$.
- Some possible candidates: $\mathbb{F}_q[[t]]$, the formal power series ring over $\mathbb{F}_q$, or $\mathbb{Z}_p$, the p-adic integer ring.

Kakeya in Non-Archimedean Local Rings, I

Question (J. Ellenberg, R. Oberlin, T. Tao, 2009)

Are there Besicovitch phenomena in $\mathbb{F}_q[[t]]^n$ *or in* $\mathbb{Z}_p^n$*?*

In other words, do there exist Besicovitch sets of measure 0 in these spaces?

Kakeya in Non-Archimedean Local Rings, I

Question (J. Ellenberg, R. Oberlin, T. Tao, 2009)

Are there Besicovitch phenomena in $\mathbb{F}_q[[t]]^n$ or in $\mathbb{Z}_p^n$?

In other words, do there exist Besicovitch sets of measure 0 in these spaces?

Theorem (E.D., M. Hablicek, 2011)

There exists a Besicovitch set of measure 0 in $\mathbb{F}_q[[t]]^n$ for each $n \geq 2$.

Proof: Explicit construction.

Kakeya in Non-Archimedean Local Rings, II

Theorem (R. Fraser, 2015)

For $n \geq 2$, there exists a Besicovitch set of measure zero over R^n for any discrete valuation ring R with finite residue field.

Fraser's construction is more analytic, involving various classes of differentiable functions.

Theorem (X. Caruso, 2016)

For $n \geq 2$, almost all Kakeya sets in R^n have Haar measure zero for any discrete valuation ring R with finite residue field.

The difference between Kakeya sets and Besicovitch sets (in Caruso's formulation) is that Kakeya sets also possess a continuity condition.

Kakeya in Non-Archimedean Local Rings, III

We can also pose the Kakeya conjecture in the local ring setting. Here is the appropriate notion of dimension:

Definition

If R is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and $\mathbb{F}_q = R/\mathfrak{m}$ finite, the **Minkowski dimension** *of a subset E of R^n is*

$$\lim_{k \to \infty} \frac{\log N(k)}{\log q^k}$$

where $N(k)$ is the size of the image of E under the map $R \to R/\mathfrak{m}^k$.

In this case, we are "covering" the set with boxes of size $1/q^k$.

Kakeya in Non-Archimedean Local Rings, IV

Conjecture (Kakeya Conjecture)

For $n \geq 2$, the Minkowski dimension of a Besicovitch set in R^n where $R = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$ is n.

Kakeya in Non-Archimedean Local Rings, IV

Conjecture (Kakeya Conjecture)

For $n \geq 2$, the Minkowski dimension of a Besicovitch set in R^n where $R = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$ is n.

We have some partial progress toward this result.

Theorem (E.D., M. Hablicek, 2011)

The Minkowski dimension of a Besicovitch set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$ is 2.

In dimensions $n \geq 3$ over these rings, the Kakeya conjecture remains open, just like over $\mathbb{R}$....

Applications of Kakeya Sets

Kakeya sets have a number of applications in wide-ranging areas:

- Harmonic analysis (Fefferman): counterexamples to some Fourier convergence results in L^p norm rely on Kakeya sets.
- Solutions to the wave equation (Wolff): certain kinds of bounds fail, with Kakeya sets giving counterexamples.
- Error-correcting codes and cryptography (Bourgain): Kakeya sets are related to certain kinds of error-correcting codes.
- Analytic number theory and additive combinatorics (Tao, Bourgain, N. H. Katz, many others): Kakeya sets are related to various sum-product problems.

Open Questions

Here are a few broad questions that are still open:

- What kinds of interactions are there between the Kakeya problems in $\mathbb{R}$, $\mathbb{F}_q$, $\mathbb{F}_q[[t]]$, and $\mathbb{Z}_p$?
- Can we use Kakeya sets in $\mathbb{F}_q[[t]]$ and $\mathbb{Z}_p$ in harmonic analysis over these rings, in a similar way to how they are used for harmonic analysis on $\mathbb{R}$?
- Can we use methods for studying the algebraic Kakeya problems on the analytic side (or vice versa)?

End of Talk

Thank you for attending my talk!