Kakeya Sets Over non-Archimedean Local Rings

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Outline

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1. Classical Kakeya problem
2. Finite-field Kakeya problem
3. Kakeya problem over local rings
4. Open questions

This is joint work with M. Hablicek (UW-Madison).

Classical Kakeya Problem

Definition

A **Kakeya set** *(or Besicovitch set) is a set of points in Euclidean space which contains a unit line segment in every direction.*

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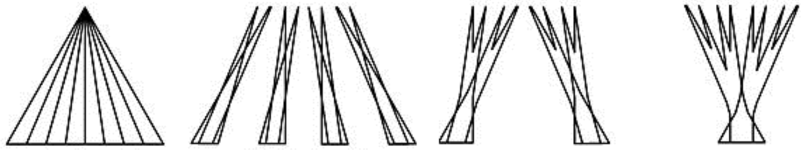
Theorem (A. Besicovitch; 1919)

For $n \geq 2$, there exists a Kakeya set of measure 0 in $\mathbb{R}^n$.



The Kakeya conjecture

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The **Minkowski dimension** *of a set K is defined to be*

$$\dim(K) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

where $N(\epsilon)$ is the number of squares of size ϵ needed to cover K.

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Conjecture (Kakeya conjecture)

Any Kakeya set in $\mathbb{R}^n$ is of Minkowski dimension n.

The conjecture is known to hold when $n = 2$ but only lower bounds are known for $n > 2$.

Kakeya sets over finite fields

Definitions:

- Let $\mathbb{F}_q$ be a finite field, and n a fixed positive integer.
- Space of interest: $S = \mathbb{F}_q^n$.
- Lines in S are of the form $\{x + sy \ : \ s \in \mathbb{F}_q, \, x, y \in S\}$.
- A direction in S is an equivalence class of y giving the same line.
- A **Kakeya set** is a set containing a line in every direction.

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Theorem (Dvir; 2008)

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $\dfrac{q^n}{n!}$ points.

In other words, a Kakeya set over a finite field always has positive measure. (This proves the Kakeya conjecture over $\mathbb{F}_q$.)

Between $\mathbb{R}$ and $\mathbb{F}_q$

Over $\mathbb{R}$ there exist Kakeya sets of measure zero, but over $\mathbb{F}_q$, there exists a hard lower bound on measure (independent of q). What's in between?

Question (J. Ellenberg, R. Oberlin, T. Tao; 2009)

Are there Besicovitch phenomena in $\mathbb{F}_q[[t]]^n$ or in $\mathbb{Z}_p^n$?

In other words, do there exist Kakeya sets of measure 0 in these spaces?

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Theorem (E.D., Hablicek; 2011)

There exists a Kakeya set of measure 0 in $\mathbb{F}_q[[t]]^n$ for each $n \geq 2$.

Proof: Explicit construction.

Preliminaries

Definitions:

- Space of interest: $S = \mathbb{F}_q[[t]]^n$.
- Lines in S are of the form $\{x + sy \ : \ s \in \mathbb{F}_q[[t]],\ x, y \in S\}$.
- A direction in S is an equivalence class of y, for which at least one coordinate is a unit.
- A **Kakeya set** is a set containing a line in every direction.
- The Haar measure μ on S is generated by the projections $\pi_k : \mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]/(t^k)$, where the measure on $(\mathbb{F}_q[[t]]/(t^k))^n$ is the probability measure.

Construction, I

Observations:

- Only need to construct a Kakeya set K of measure zero in $\mathbb{F}_q[[t]]^2$, then just take $K \times \mathbb{F}_q[[t]]^{n-2}$ in general.
- In $\mathbb{F}_q[[t]]^2$, only need to construct a set H containing all lines with direction vectors $(1, b)$; then $K = \{(x, y) : (x, y) \text{ or } (y, x) \in H\}$ is Kakeya.

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- In $\mathbb{F}_q[[t]]^2$, only need to construct a set H containing all lines with direction vectors $(1, b)$; then $K = \{(x, y) : (x, y) \text{ or } (y, x) \in H\}$ is Kakeya.
- For any odd map $* : \mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]$, the set $H_* = \{(x, y) : ax + y = a^* \text{ for some } a \in \mathbb{F}_q[[t]]\}$ will contain a line with direction vectors $(1, b)$.
- Reason: For any $b \in \mathbb{F}_q[[t]]$, the points $(x, y) = (0, -b^*) + s(1, b)$ are in H_*.

Construction, II

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Solution:

- For any $a \in \mathbb{F}_q[[t]]$, let a_i denote the coefficient of t^i.
- For any $a \in \mathbb{F}_q[[t]]$, define $a^* \in \mathbb{F}_q[[t]]$ by setting

$$a_i^* = \begin{cases} 0 & \text{if } i = 2^k - 2 \text{ for some natural number } k, \\ a_{i+1} & \text{otherwise.} \end{cases}$$

Proposition

With notation as previous, the set

$$H := \{(x, y) \in \mathbb{F}_q[[t]]^2 : \ ax + y = a^* \text{ for some } a \in \mathbb{F}_q[[t]]\}$$

contains a line with direction vector $(1, b)$ for each $b \in \mathbb{F}_q[[t]]$, and has measure 0.

Construction, III: Sketch of Proof

The point (x, y) is in H if and only if there exists an $a = a_0 + a_1 t + a_2 t^2 + \cdots$ satisfying the system

$$
\begin{aligned}
a_0 x_0 + y_0 &= 0, && [0] \\
a_1 x_0 + a_0 x_1 + y_1 &= a_2, && [1] \\
a_2 x_0 + a_1 x_1 + a_0 x_2 + y_2 &= 0, && [2] \\
&\vdots \\
a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n + y_n &= a_n^*, && [n] \\
&\vdots
\end{aligned}
$$

Construction, III: Sketch of Proof

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$$\begin{aligned} a_0 x_0 + y_0 &= 0, && [0] \\ a_1 x_0 + a_0 x_1 + y_1 &= a_2, && [1] \\ a_2 x_0 + a_1 x_1 + a_0 x_2 + y_2 &= 0, && [2] \\ &\vdots \\ a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n + y_n &= a_n^*, && [n] \\ &\vdots \end{aligned}$$

Now count solutions to equations [0]-[n] as n grows, and show that the measure of the set of points (x, y) such that there exists an a satisfying [0]-[n] tends to 0 as n grows.

Minkowski Dimension, I

We can also pose the Kakeya conjecture over the local ring setting.

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Definition

The **Minkowski dimension** *of a subset E of $\mathbb{F}_q[[t]]^n$ is*

$$\lim_{k\to\infty} \frac{\log |E_k|}{\log q^k}$$

where E_k is the image of E under $\mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]/(t^k)$.

(We have an analogous definition for subsets of $\mathbb{Z}_p^n$.)

Conjecture (Kakeya Conjecture)

For $n \geq 2$, the Minkowski dimension of a Kakeya set in R^n where $R = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$ is n.

Minkowski Dimension, II

Theorem (E.D., M. Hablicek; 2011)

The Minkowski dimension of a Kakeya set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$ is 2.

In dimensions $n \geq 3$ over these rings, the Kakeya conjecture remains open.

Minkowski Dimension, II

Theorem (E.D., M. Hablicek; 2011)

The Minkowski dimension of a Kakeya set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$ is 2.

In dimensions $n \geq 3$ over these rings, the Kakeya conjecture remains open. Key result in proof:

Proposition

Let E be a Kakeya set in R^2 where $R = \mathbb{F}_q[t]/t^k$ or $\mathbb{Z}/p^k\mathbb{Z}$. Then $|E| \geq \dfrac{|R|^2}{2k}$.

This proposition is a sharpening of a counting result of Ellenberg-Oberlin-Tao.

Open Questions

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Does a Kakeya set of measure 0 exist in $\mathbb{Z}_p^2$? (Or other rings?)

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Kakeya sets over $\mathbb{R}$ are the central ingredient in a theorem on L^p spaces in classical analysis:

Theorem (C. Fefferman; 1971)

The truncated Fourier operator T on $L^p(\mathbb{R}^n)$, defined by $\hat{T}f(x) = \chi_p(x)\hat{f}(x)$, where χ_p is the characteristic function of the unit ball, is bounded only on L^2.

If a Kakeya set of measure zero exists over $\mathbb{Z}_p$, it may be possible to obtain a similar theorem over $\mathbb{Z}_p$.

End of Talk

Thank you!