The Kakeya Problem

Evan P. Dummit

Arizona State University

November 15, 2017

The Kakeya Needle Problem, I

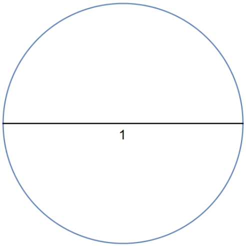
Definition (S. Kakeya, 1917)

A ***Kakeya needle set*** *is a subset of the plane inside which it is possible to rotate a needle of length 1 completely around.*

The Kakeya Needle Problem, II

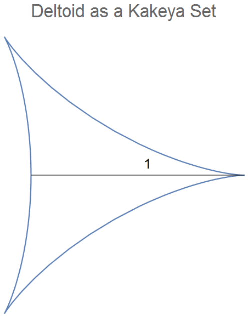
An example of a Kakeya set: a circle of diameter 1 (area $\pi/4$):

Circle as a Kakeya Set



The Kakeya Needle Problem, III

Another example of a Kakeya set: a deltoid (area $\pi/8$):



The Kakeya Needle Problem, IV

Question

What is the minimum area of a Kakeya needle set?

It was originally believed that the deltoid example (of area $\pi/8$) was the smallest possible Kakeya set. But....

The Kakeya Needle Problem, IV

Question

What is the minimum area of a Kakeya needle set?

It was originally believed that the deltoid example (of area $\pi/8$) was the smallest possible Kakeya set. But....

Theorem (A. Besicovitch, 1919)

There exists a Kakeya needle set in the plane having arbitrarily small area.

The Kakeya Needle Problem, V

Basic idea for constructing a Kakeya set of small area:

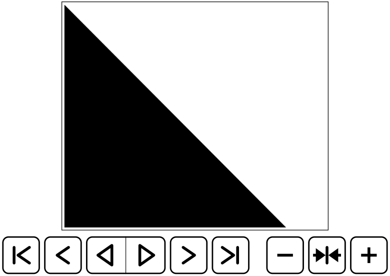
The Kakeya Needle Problem, V

Basic idea for constructing a Kakeya set of small area:

- Start with a simple Kakeya set.
- Slice up the set into pieces.
- Slide the the pieces together so that they overlap a lot.
- Repeat steps 2-3 until the set is arbitrarily small.

The Kakeya Needle Problem, VI

Here's an example of how this "slicing and sliding" procedure works:



The Kakeya Needle Problem, VII: The Force Awakens

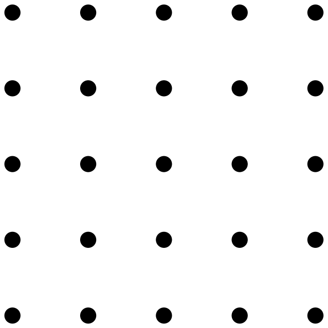
There are many ways to tweak this problem to give us new ones. Here are some:

- We could relax the requirement of having an actual needle moving through space, and just ask about sets that contain a unit segment in every possible direction.
- Instead of working in the plane, we could work in 3-dimensional space, and ask about (surface) areas or volumes.
- We could use shapes other than a straight line segment, like a bent segment or even a curve.

Try to think of your own version of this problem!

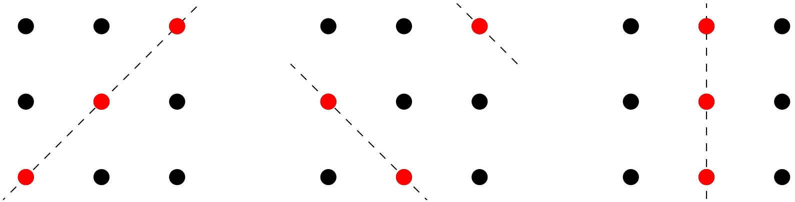
The Finite Kakeya Problem, I

Let's now look at the Kakeya problem "modulo p" (where p is a prime number). Instead of the entire plane, we use a $p \times p$ grid of lattice points:



The Finite Kakeya Problem, II

Lines "wrap around" the edges of the grid:



The Finite Kakeya Problem, III

We need a few more facts about lines in our $p \times p$ grid modulo p:

- Each line contains p points.
- Algebraically, a line will be an equation of the form $ax + by \equiv c \pmod{p}$ for some integers a, b, and c.
- Any non-vertical line can be put into the usual form $y \equiv mx + b \pmod{p}$.
- Just like regular lines in the plane, these lines have a slope (which is the value m above, or ∞ for a vertical line).
- Two lines point in the same direction (i.e., are "parallel") if they have the same slope.
- There are $p + 1$ possible directions: slope 0, slope 1, ... , slope $p - 1$, and slope ∞.

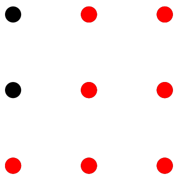
The Finite Kakeya Problem, IV

Here is the "mod-p" Kakeya problem:

Definition

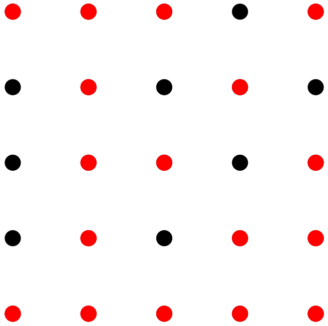
A **Kakeya set** *is a set of points in the $p \times p$ which contains a line in every direction.*

By "contains a line" we mean "contains the p points on the line". Here is an example of a 3×3 Kakeya set:



The Finite Kakeya Problem, V

Here is an example of a Kakeya set for the 5×5 grid:



The Finite Kakeya Problem, VI

So how small can a Kakeya set in the $p \times p$ grid be?

The Finite Kakeya Problem, VI

So how small can a Kakeya set in the $p \times p$ grid be?

Proposition

Any Kakeya set in the $p \times p$ grid contains at least $\frac{1}{2}p^2$ points.

The Finite Kakeya Problem, VI

So how small can a Kakeya set in the $p \times p$ grid be?

Proposition

Any Kakeya set in the $p \times p$ grid contains at least $\frac{1}{2}p^2$ points.

Proof: The first line has p points, the second adds at least $p - 1$ new points, the third adds at least $p - 2$ more, and so on. Therefore, the total number of points in the Kakeya set is at least

$$p + (p-1) + (p-2) + \ldots + 2 + 1 = \frac{p(p+1)}{2} > \frac{1}{2}p^2$$

points, as claimed. $\square$

The Finite Kakeya Problem, VII

How can we generalize this problem? One way: work with a "grid" in 3 or higher-dimensional space.

Conjecture (Finite-Field Kakeya Conjecture)

Any n-dimensional Kakeya set modulo p contains at least $c_n p^n$ points, for some constant $c_n > 0$.

Originally posed by T. Wolff in 1999. This problem seemed extremely hard!

The Finite Kakeya Problem, VII

How can we generalize this problem? One way: work with a "grid" in 3 or higher-dimensional space.

Conjecture (Finite-Field Kakeya Conjecture)

Any n-dimensional Kakeya set modulo p contains at least $c_n p^n$ *points, for some constant* $c_n > 0$.

Originally posed by T. Wolff in 1999. This problem seemed extremely hard!

Theorem (G. Mockenhaupt, T.Tao, 2004)

Any n-dimensional Kakeya set modulo p contains at least $c_n p^{(4n+3)/7}$ *points, for a constant* $c_n > 0$.

The Finite Kakeya Problem, VIII

But, it turns out, this problem has an extremely clever and very short solution using nothing more than polynomials:

Theorem (Dvir; 2008)

Any n-dimensional Kakeya set modulo p contains at least $\binom{n+q-1}{n} \geq \frac{p^n}{n!}$ *points.*

In other words, a Kakeya set always contains a positive proportion of the points, even when the grid size gets very large.

The Finite Kakeya Problem, IX

There are lots of other ways to generalize this problem. For example, we could try working with an $n \times n$ grid of points where n is not prime. But be warned: lines in this setting behave more strangely!

In fact, the precise answers in this case are not known! (Perhaps you might have a good idea....)