Number Theory (part 7): Elliptic Curves (by Evan Dummit, 2021, v. 1.01)

Contents

7 Elliptic Curves

In this chapter, we will introduce elliptic curves and describe some aspects of their relationship to number theory. Elliptic curves have a long and interesting history, and their study involves elements from most of the major disciplines of mathematics: algebra, geometry, analysis, number theory, topology, and even logic. Particularly, elliptic curves appear in the proofs of many deep results in mathematics: for example, they are a central ingredient in the proof of Fermat's Last Theorem, which states that there are no positive integer solutions to the equation $x^n + y^n = z^n$ for any integer $n \geq 3$.

Our goals are fairly modest in comparison, so we will begin by outlining the basic algebraic and geometric properties of elliptic curves and motivate the group law, which establishes that the rational points on an elliptic curve have the structure of an abelian group. We will then study the behavior of elliptic curves modulo p: owing to the fact that both sets are finite abelian groups, there are many analogies between the structure of the points on an elliptic curve modulo p and the integers modulo n. Our goal is to explore these analogies and then to use them to convert certain cryptosystems and factorization algorithms that rely on modular arithmetic to ones that rely on elliptic curves.

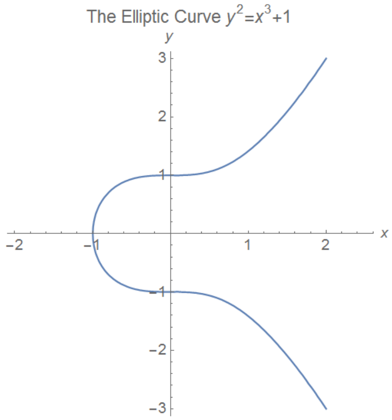
We then discuss some other results about rational and integral points on elliptic curves; the proofs of many of these results are far beyond our scope, but we can still apply them to discuss some applications of elliptic curves to Diophantine equations, such as the famous congruent number problem.

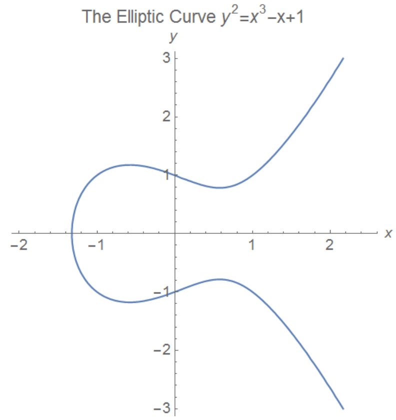
7.1 Elliptic Curves and the Addition Law

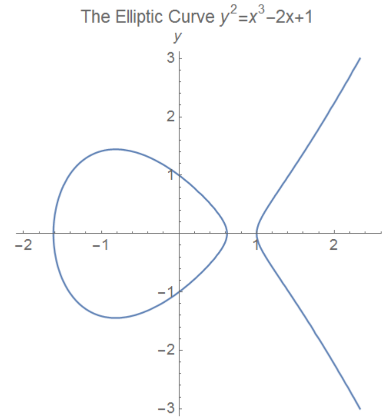
- In this section, we will outline the basic features of elliptic curves and establish the group law.

7.1.1 Cubic Curves, Weierstrass Form, Singular and Nonsingular Curves

- In elementary coordinate geometry, one begins by studying the behavior of lines in the plane, which have the general equation $ax + by + c = 0$, and then afterwards studies more general quadratic curves (the conic sections) having the general equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$.
  - In each case, we often perform simple algebraic manipulations and changes of variable to put the equations into a more standard form.
  - For example, if $b \neq 0$, we can rewrite the equation $ax + by + c = 0$ as $y = (-a/b)x + (-c/b)$, which for $m = -a/b$ and $b_1 = -c/b$ has the more familiar form $y = mx + b_1$.
  - Similarly, if $a \neq 0$, we can perform a change of variable $x_1 = y + (b/(2a))x$ in the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ to remove the cross term bxy: we eventually obtain an equation of the form $ax_1^2 + c_1y^2 + d_1x_1 + e_1y + f_1 = 0$ for new coefficients c_1, d_1, e_1, f_1.
  - We can then complete the square in both x_1 and y (again, assuming certain coefficients are nonzero) by setting $x_2 = x_1 + d_1/(2a_1)$ and $y_2 = y + e_1/(2c_1)$. Eventually we will obtain an equation having the much simpler $ax_2^2 + c_1y_2^2 + f_2 = 0$.
  - If we abuse notation by dropping the subscripts, we see that essentially every conic can be put into the form $ax^2 + cy^2 + f = 0$ after changing coordinates.
- Our goal now is to study cubic curves in the plane, which have the general form $ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j = 0$.
  - Like in the case of quadratic curves above, we can perform a series of changes of variable to reduce the general form to a simpler one.
  - We will not give the full details of the procedure, as it is rather complicated.
  - Instead, we will summarize matters by saying that as long as the equation is actually cubic (i.e., it is not the case that all of a, b, c, d are zero), then the general equation above can always be transformed using rational changes of variable into one of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, for appropriate coefficients a_1, a_2, a_3, a_4, a_6.
- Definition: An elliptic curve E over a field K is a curve having an equation of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, for appropriate coefficients a_1, a_2, a_3, a_4, a_6 in K. This expression is called the Weierstrass form of E.
  - Note: We will generally restrict our attention to the situation where K is one of the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of integers modulo p.
  - This expression is not the simplest possible one: as long as the characteristic of K is not 2 or 3, we can simplify it by completing the square in y and completing the cube in x.
  - Explicitly, if we set $y' = y + (a_1/2)x + (a_3/2)$ and $x' = x + (a_2/3)$, we can reduce the Weierstrass equation above to one of the form $(y')^2 = (x')^3 + A(x') + B$.
  - An elliptic curve having an equation of the form $y^2 = x^3 + Ax + B$ is sometimes said to be in "reduced" Weierstrass form.
  - This reduced form is much more amenable for computations, and (in fact) it is nearly unique: the only change of variables that preserves it is one of the form $x = u^2x'$, $y = u^3y'$ for some nonzero u, from which we see that $A = u^4A'$ and $B = u^6B'$.
- Here are the graphs of the elliptic curves $y^2 = x^3 + 1$, $y^2 = x^3 - x + 1$, and $y^2 = x^3 - 2x + 1$ over $\mathbb{R}$:



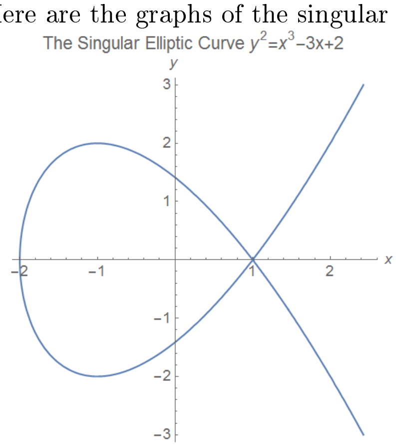


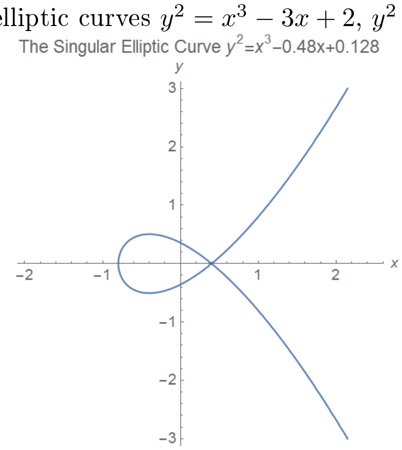


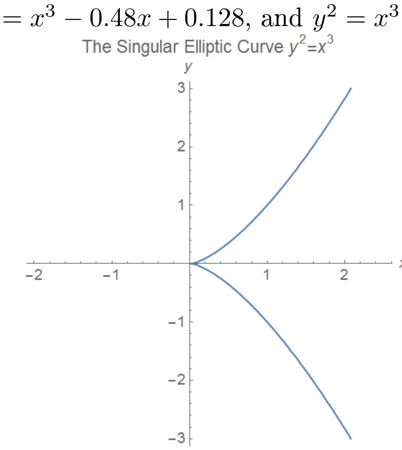
- Note that elliptic curves are not ellipses! The reason for the similar name is that if one wants to compute the arclength of an ellipse (an elliptic integral), a few changes of variable will transform the resulting integral into one of the general form $\int \frac{1}{\sqrt{x^3+Ax+B}}\,dx$. Upon setting $y = \sqrt{x^3+Ax+B}$, we see that this elliptic integral is rather naturally related to the curve $y^2 = x^3 + Ax + B$.
- In general, we can see that the graph of an elliptic curve $y^2 = x^3 + Ax + B$ will always be symmetric about the x-axis, since if (x, y) satisfies the equation then so does $(x, -y)$.
- By using this observation and invoking the implicit function theorem, it can be shown that the graph of an elliptic curve will have either one or two components depending on the values of the coefficients: it will have two components when the polynomial $x^3 + Ax + B$ has three distinct real roots, and it will have one component otherwise.
- Notice also that the tangent line at each crossing of the x-axis is vertical for each curve above. Using implicit differentiation, we can compute $y' = \frac{3x^2+A}{2y}$: thus, we see that $y' = \infty$ when y is zero, provided that $3x^2 + A$ is not also zero. This behavior can only occur when $x^3 + Ax + B$ has a root in common with its derivative $3x^2 + A$, which is in turn equivalent to saying that $x^3 + Ax + B$ has a double root.

- Definition: If the polynomial $x^3 + Ax + B$ has a repeated root, we say that the elliptic curve $y^2 = x^3 + Ax + B$ is singular. Otherwise (if the roots are distinct) we say the elliptic curve is nonsingular. A curve is singular if and only if its discriminant $\Delta = -16(4A^3 + 27B^2)$ is zero.
  - The second statement follows from the observations above: the polynomial $x^3 + Ax + B$ has a repeated root if and only if it has a root in common with its derivative $3x^2 + A$. This occurs precisely when $x^2 = -A/3$, from which we see that $x(2A/3) + B = 0$ so $x = -3B/(2A)$: then substituting for x yields $\Delta = 0$ almost immediately.
  - Remark: The presence of the constant -16 is superfluous here, but there is also a definition of Δ in terms of the original coefficients a_1, a_2, a_3, a_4, a_6 for a general Weierstrass form. To avoid having denominators in that expression, we end up needing an extra factor of -16 in the one we gave above.

- Here are the graphs of the singular elliptic curves $y^2 = x^3 - 3x + 2$, $y^2 = x^3 - 0.48x + 0.128$, and $y^2 = x^3$:







- The singular point (i.e., the point where the curve is nondifferentiable) on the first two curves is where the curve crosses itself. This type of singularity is known as a node, and will occur when the polynomial $x^3 + Ax + B$ has a double root.
- The singular point on the third curve is the cusp at the origin $(0,0)$. This type of singularity will occur when the polynomial $x^3 + Ax + B$ has a triple root, which can only happen when $A = B = 0$.

- In general, singular elliptic curves tend to have unusual properties relative to nonsingular curves. We will therefore exclude singular elliptic curves and speak only of nonsingular elliptic curves from this point on.

7.1.2 The Addition Law

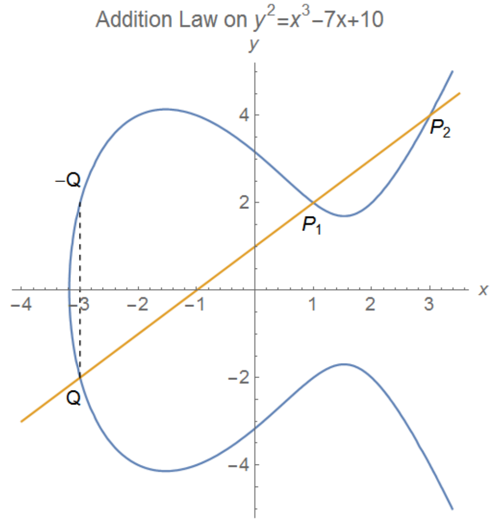
- The key property of elliptic curves that make them so useful is that, if we have two points that lie on the curve, we can use them to construct a third point on the curve.
  - Explicitly, suppose $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two distinct points on an elliptic curve E: $y^2 = x^3 + Ax + B$.
  - Draw the line through P_1 and P_2: we claim that this line L must intersect E in a third point Q.
  - To see this, suppose the line through P_1 and P_2 has equation $y = mx + b$. (We are tacitly excluding the possibility that the line is vertical, but we will come back to this case in a moment.)
  - Then the intersection points between L and E are the solutions to the system $y = mx + b$ and $y^2 = x^3 + Ax + B$. Equivalently, we must solve $(mx+b)^2 = x^3+Ax+B$, or $x^3+(-m^2)x^2+(A-2mb)x+(B-b^2) = 0$.
  - However, we already know that this cubic has the two roots $x = x_1$ and $x = x_2$, so it must have a third root: this gives us the third point Q we wanted.

- Once we construct a third point on an elliptic curve this way, we might try to find more points.
  - If we try this procedure directly using our points P_1, P_2, and Q, however, we will not get anywhere: the line through any of these two points intersects the elliptic curve at the other point.
  - However, we can also exploit the vertical symmetry of the curve to make new points: if $P = (x, y)$ lies on the curve, then the point $-P = (x, -y)$ also lies on the curve.
  - If we combine these two procedures, we can often generate many points on the curve starting from just two.

- Definition (Group Law I): If P_1 and P_2 are two distinct points on the elliptic curve $E : y^2 = x^3 + Ax + B$, let $Q = (x', y')$ be the third intersection point of E with the line L joining P_1 and P_2. We define the sum $P_1 + P_2$ to be the point $-Q = (x', -y')$.
  - Important Note: The sum $P_1 + P_2$ is *not* the pointwise coordinate sum of P_1 and P_2!
  - It is not immediately clear why we define the sum of two points to be the reflection of Q rather than Q itself. This will become clearer in a moment.
  - Note that if we attempt to add two points which are vertical reflections of one another on the graph of $y^2 = x^3 + Ax + B$, the resulting line will not intersect the curve again.
  - To remedy this, we declare that the curve also includes a point at ∞ (which we denote simply as ∞) that we consider as lying on any vertical line.
  - The reason for this declaration is that we want to be able to calculate the sum of any two points on the curve.

- Example: Given the points $P_1 = (1, 2)$ and $P_2 = (3, 4)$ on the elliptic curve $y^2 = x^3 - 7x + 10$, find the sums $P_1 + P_2$ and $(P_1 + P_2) + P_2$.

- It is easy to verify that both points lie on the curve. Here is a plot of the curve and the line $y = x + 1$ through the two points:



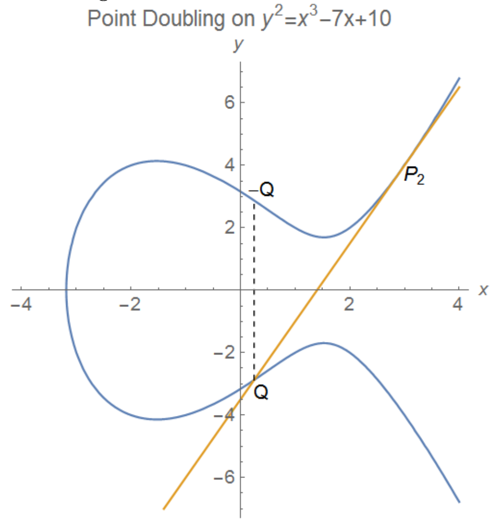
- Now we find the exact coordinates of Q.
- The point lies on the intersection of $y = x+1$ and $y^2 = x^3 - 7x + 10$, so $(x+1)^2 = x^3 - 7x + 10$.
- This equation is equivalent to $x^3 - x^2 - 9x + 9 = 0$, which factors as $(x-1)(x-3)(x+3) = 0$. Then the x-coordinate of Q is -3 so $Q = (-3, -2)$.
- Thus, the sum $P_1 + P_2$ is the vertical reflection of Q, which is $\boxed{(-3, 2)}$.
- To find the sum $(P_1 + P_2) + P_2$ we perform a similar procedure: the line through $P_1 + P_2$ and P_2 has equation $y = \frac{1}{3}x + 3$.
- Then we must solve $(\frac{1}{3}x + 3)^2 = x^3 - 7x + 10$, or $x^3 - \frac{1}{9}x^2 - 9x + 1 = 0$.
- Factoring yields $(x - \frac{1}{9})(x+3)(x-3) = 0$, so $Q' = (\frac{1}{9}, \frac{82}{27})$, and thus $(P_1 + P_2) + P_2 = \boxed{(\frac{1}{9}, -\frac{82}{27})}$.

- Now that we have defined addition, a natural question is whether we can add a point to itself.
  - It is straightforward to see from our definition that if P_1 and P_2 are distinct points, then $P_1 + P_2$ is a continuous function of the coordinates of the points.
  - If we are working over $\mathbb{R}$, we could therefore define the addition $P + P$ to be the limit as $P_1 \to P$ of sums $P + P_1$. Geometrically, the lines used in the construction also have a limit as $P \to P_1$: they approach the tangent line to the curve E at the point P.
  - Thus, a natural way to define $P + P$ is to let L be the tangent line to E at P, and then take Q to be the third point of intersection of L with E.

- Definition (Group Law II): If P is any point on the elliptic curve $E : y^2 = x^3 + Ax + B$, let $Q = (x', y')$ be the third intersection point of E with the tangent line L to E at P. We define the sum $P + P$ to be the point $-Q = (x', -y')$.

- Example: Given the points $P_1 = (1, 2)$ and $P_2 = (3, 4)$ on the elliptic curve $y^2 = x^3 - 7x + 10$, find the sums $P_2 + P_2$ and $(P_1 + P_2) + P_2$.
  - Differentiating implicitly yields $2yy' = 3x^2 - 7$ so that $y' = (3x^2 - 7)/(2y)$. Thus, the tangent line to E at P_2 has slope $\frac{5}{2}$ and its equation is $y = \frac{5}{2}x - \frac{7}{2}$.

- Here is a plot of the curve and the tangent line at P_2:



- The point Q lies on the intersection of $y = \frac{5}{2}x - \frac{7}{2}$ and $y^2 = x^3 - 7x + 10$, so $(\frac{5}{2}x - \frac{7}{2})^2 = x^3 - 7x + 10$.
- This equation is equivalent to $x^3 - \frac{25}{4}x^2 + \frac{21}{2}x - \frac{9}{4} = 0$, which factors as $(x - \frac{1}{4})(x-3)(x-3) = 0$.
- Then the x-coordinate of Q is $1/4$ so $Q = (\frac{1}{4}, -\frac{23}{8})$, and so $P_2 + P_2 = \boxed{(\frac{1}{4}, \frac{23}{8})}$.
- To find the sum $P_1 + (P_2 + P_2)$ we then find the sum of $P_1 = (1, 2)$ with $(\frac{1}{4}, \frac{23}{8})$. The line through these points is $y = -\frac{7}{6}x + \frac{19}{6}$.
- Then we must solve $(-\frac{7}{6}x + \frac{19}{6})^2 = x^3 - 7x + 10$, which has solutions $x = \frac{1}{9}, \frac{1}{4}, 1$.
- Then $Q' = (\frac{1}{9}, \frac{82}{27})$, and thus $P_1 + (P_2 + P_2) = \boxed{(\frac{1}{9}, -\frac{82}{27})}$.

- Note that in the previous two examples, we computed $(P_1 + P_2) + P_2 = (\frac{1}{9}, -\frac{82}{27}) = P_1 + (P_2 + P_2)$, and so we see in this case that the addition law is actually associative. Much more is true:
- Theorem (Group Law): If K is any field and E is any elliptic curve defined over K, then for any points P, P_1, P_2, and P_3 on E, the following are true:

  1. The addition law is commutative: $P_1 + P_2 = P_2 + P_1$.
  2. The addition law is associative: $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$.
  3. The point at ∞ is a two-sided identity: $P + \infty = P = \infty + P$.
  4. The point P has a two-sided inverse $-P$: $P + (-P) = \infty = (-P) + P$.

  - A more concise way of phrasing this statement is to say that the set of points on E (including the point at ∞) forms an abelian group.
  - We will give arguments for an elliptic curve of the form $y^2 = x^3 + Ax + B$, but the theorem holds in full generality for any elliptic curve.
  - Proof (1): The first part is immediate from the geometric definition we have given since the line used in computing $P_1 + P_2$ and $P_2 + P_1$ is the same in each case.
  - Proof (2): This part, which is the only nontrivial result in this theorem, can be done with a lengthy numerical computation using explicit formulas for the addition law (see below). We omit the details.
  - Proof (3): Consider the sum $P + \infty$. The line passing through P and ∞ is the vertical line through P which also intersects E at the point $-P$. Then by the geometric definition, $P + \infty = -(-P) = P$.

- Proof (4): Consider the sum $P+(-P)$. The line passing through P and $-P$ is a vertical line, so the other point on it is ∞. The reflection of ∞ is also ∞, so $P+(-P)=\infty$.

- For convenience in doing numerical computations, we will also write down the general formula for the addition law on any curve:

- Proposition (Explicit Group Law): Let $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ be points on the elliptic curve $E: y^2=x^3+Ax+B$. Then $P_1+P_2=(x_3,y_3)$ where $x_3=m^2-x_1-x_2$ and $y_3=-m(x_3-x_1)-y_1$, with $m=\begin{cases}(y_2-y_1)/(x_2-x_1) & \text{if } P_1\neq P_2\\ (3x_1^2+A)/(2y_1) & \text{if } P_1=P_2\end{cases}$. If m is infinite, then $P_1+P_2=\infty$.
  - We will remark that the addition formula is rational, in the sense that the result is always a rational function of the inputs. In particular, the sum of two points whose coordinates lie in a field K will also lie in K.
  - We will also remark that there are formulas for the addition law on a more general elliptic curve $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$, but we will not need them.
  - Proof: If $P_1\neq P_2$ then the line joining P_1 and P_2 has equation $y-y_1=m(x-x_1)$ where $m=\dfrac{y_2-y_1}{x_2-x_1}$.
  - We therefore obtain the equation $(mx-mx_1+y_1)^2=x^3+Ax+B$, which has the form $x^3-m^2x^2+Cx+D=0$ for appropriate constants C and D.
  - The polynomial $x^3-m^2x^2+Cx+D$ must factor as $(x-x_1)(x-x_2)(x-x_3)$, so upon multiplying out we see that $x_1+x_2+x_3=m^2$. This yields the stated value of x_3, and then $y_3=m(x_3-x_1)+y_1$ (where we have multiplied by -1 to account for the vertical reflection).
  - If $P_1=P_2$ then everything is the same, except instead m is the slope of the tangent line at P_1. By implicit differentiation, we see that $2yy'=3x^2+A$ so $m=\dfrac{3x_1^2+A}{2y_1}$ here, as claimed.

7.1.3 Elliptic Curves Modulo p

- We have primarily dealt with elliptic curves over the real numbers, but an important part of the general theory requires studying elliptic curves modulo p, where p is prime. We will take $p\geq 5$ to be a prime throughout the remainder of this section.
  - We will discuss elliptic curves modulo nonprime integers when we discuss factorization algorithms.
- All of our analysis of elliptic curves carries into this setting essentially verbatim: in particular, the properties of the addition law and the algebraic formulas remain the same, though we must rely on algebra rather than geometric intuition.
  - The only possible difficulty is that if we want to work in a field of "characteristic 2" (in which $2=0$) or "characteristic 3" (in which $3=0$), we will need to use the general Weierstrass form $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$ rather than the reduced Weierstrass form $y^2=x^3+Ax+B$.
  - As we showed earlier, an elliptic curve $y^2=x^3+Ax+B$ is nonsingular modulo p precisely when its discriminant $\Delta=-16(4A^3+27B^2)$ is nonzero modulo p.
  - In particular, we can see that a curve of this form will always be singular modulo 2.
  - More generally, if we have any elliptic curve, the primes p for which the curve is singular mod p (the primes of "bad reduction") are precisely the primes dividing the discriminant Δ.
- Example: If $P_1=(1,3)$ and $P_2=(0,2)$ on the elliptic curve $y^2=x^3+4x+4$ modulo 5, find P_1+P_2 and P_1+P_1.
  - We simply apply the appropriate formulas: adding $Q_1=(x_1,y_1)$ to $Q_2=(x_2,y_2)$ produces (x_3,y_3) where $x_3=m^2-x_1-x_2$ and $y_3=-m(x_3-x_1)-y_1$, and $m=\begin{cases}(y_2-y_1)/(x_2-x_1) & \text{if } Q_1\neq Q_2\\ (3x_1^2+A)/(2y_1) & \text{if } Q_1=Q_2\end{cases}$.

- With $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (0, 2)$ we obtain $m = \dfrac{2-3}{0-1} = 1$, so $x_3 = 0$ and $y_3 = -1(0-1)-3 = 3$, so $P_1 + P_2 = \boxed{(0,3)}$.
- Likewise, with $(x_1, y_1) = (x_2, y_2) = (1, 3)$ we obtain $m = \dfrac{3+4}{2 \cdot 3} = 2$, so $x_3 = 2$ and $y_3 = -2(2-1)-3 = 0$, so $P_1 + P_1 = \boxed{(2,0)}$.

- Since there are only finitely many pairs of numbers modulo p, any elliptic curve E will have only finitely many points modulo p, and so we can in principle write them all down (at least if p is small).
  - Usually, the easiest procedure for doing this is to try plugging in each possible value of x and then try to compute the square root of $x^3 + Ax + B$ to find the value of y.
  - In our count, we also include the point at ∞ on our list.
- Example: Construct an addition table for the (nonsingular) elliptic curve $y^2 = x^3 + 4x + 4$ modulo 3.
  - First, we find all the points by plugging in each of the possible x and computing the necessary square roots. We obtain

x	0	1	2
$x^3 + 4x + 4$	1	0	2
y	± 1	0	n/a

  and so there are 4 points on the curve modulo 3: $(0, 1)$, $(0, 2)$, $(1, 0)$, and ∞.
  - We can now compute all of the sums using the algebraic formulas:

$+$	∞	$(0,1)$	$(0,2)$	$(1,0)$
∞	∞	$(0,1)$	$(0,2)$	$(1,0)$
$(0,1)$	$(0,1)$	$(1,0)$	∞	$(0,2)$
$(0,2)$	$(0,2)$	∞	$(1,0)$	$(0,1)$
$(1,0)$	$(1,0)$	$(0,2)$	$(0,1)$	∞

- Example: Verify that the elliptic curve $y^2 = x^3 + 4x + 4$ is nonsingular mod p and then find all the points on the curve mod p, where $p = 5$, 7, 11, and 13.
  - For the nonsingularity part, we compute the discriminant $\Delta = -16 \cdot 688 = -2^8 \cdot 43$. Since none of 5, 7, 11, 13 divide the discriminant, the curve is nonsingular for each of these moduli.
  - To count the points, we plug in each possible value of x mod p and then try to compute the square root of $x^3 + Ax + B$.
  - Modulo 5, we obtain

x	0	1	2	3	4
$x^3 + 4x + 4$	4	4	0	3	4
y	± 2	± 2	0	n/a	± 2

  and so there are $\boxed{8}$ points modulo 5: $(0, 2)$, $(0, 3)$, $(1, 2)$, $(1, 3)$, $(2, 0)$, $(4, 2)$, $(4, 3)$, and ∞.
  - Modulo 7, we obtain

x	0	1	2	3	4	5	6
$x^3 + 4x + 4$	4	2	6	1	0	2	6
y	± 2	± 3	n/a	± 1	0	± 3	n/a

  and so there are $\boxed{10}$ points modulo 7: $(0, 2)$, $(0, 5)$, $(1, 3)$, $(1, 4)$, $(3, 1)$, $(3, 6)$, $(4, 0)$, $(5, 3)$, $(5, 4)$, and ∞.
  - Modulo 11, we obtain

x	0	1	2	3	4	5	6	7	8	9	10
$x^3 + 4x + 4$	4	9	9	10	7	6	2	1	9	10	10
y	± 2	± 3	± 3	n/a	n/a	n/a	n/a	± 1	± 3	n/a	n/a

  and so there are $\boxed{11}$ points modulo 11: $(0, \pm 2)$, $(1, \pm 3)$, $(2, \pm 3)$, $(7, \pm 1)$, $(8, \pm 3)$, and ∞.

- Modulo 13, we obtain

x	0	1	2	3	4	5	6	7	8	9	10	12
x^3+4x+4	4	9	7	4	6	6	10	11	2	2	4	1
y	± 2	± 3	n/a	± 2	n/a	n/a	± 6	n/a	n/a	n/a	± 2	± 1

and so there are $\boxed{13}$ points modulo 13: $(0, \pm 2)$, $(1, \pm 3)$, $(3, \pm 2)$, $(6, \pm 6)$, $(10, \pm 2)$, $(12, \pm 1)$, and ∞.

- Notice that the number of points on the elliptic curve E modulo p in the example above was fairly close to p for each value we tested. It turns out that this is no accident:

- Theorem (Hasse): Let E be a nonsingular elliptic curve defined over a finite field with q elements. Then the number of points $N_q(E)$ on E whose entries are in K satisfies $|N_q(E) - q - 1| \le 2\sqrt{q}$.
  - Remark: A stronger result holds for singular curves: the number of points on a singular elliptic curve (including the singular point itself) is always either p, $p+1$, or $p+2$ depending on the type of singularity.
  - Remark: When p is a prime, it is known that each of the possible integral values of $N_p(E)$ satisfying the inequality $|N_p(E) - p - 1| \le 2\sqrt{p}$ actually does occur as the number of points on some elliptic curve mod p.
  - We will not prove this result, which is usually known as the Hasse bound, as it requires more advanced methods than those we will develop[1]. However, we can give a bit of motivation by finding the number of points we should expect to be on an elliptic curve modulo p.
  - For each of the p possible values of x, there are either 2, 1, or 0 possible values of y, according to whether x is a nonzero square, zero, or a nonsquare. It can be shown that (when p is an odd prime) there are $(p-1)/2$ nonzero squares modulo p, so the expected number of values of y for any particular x is $\dfrac{1}{p}\left[2 \cdot \dfrac{p-1}{2} + 1 \cdot 1 + 0 \cdot \dfrac{p-1}{2}\right] = \dfrac{1}{p}[p-1+1] = 1$.
  - Since there are p possible x, the expected number of points (x, y) is $p \cdot 1 = p$. Together with the point at ∞, this gives $p+1$ points on the curve E.
  - Trivially, we can see that $1 \le N_p(E) \le 2p+1$: each value of x contributes at most 2 values of y, and the point at ∞ always counts. We can rearrange this inequality to read $|N_p(E) - p - 1| \le p$.
  - Hasse's theorem is then a strengthening of this inequality: it says that the actual number of points on the curve is comparatively close to the expected number of points, with the upper bound p on the difference replaced with the (comparatively much smaller) bound $2\sqrt{p}$.

7.1.4 Orders of Points

- Our goal now is to set up a rough analogy between the structure of the points on an elliptic curve modulo p under addition and the units modulo n under multiplication.
  - Ultimately, the similarities between the structure of points on an elliptic curve modulo p and the integers modulo n stem from the fact that the set of points on an elliptic curve modulo p under addition is a finite abelian group, as is the set of units modulo n.

- Our first goal is to define the order of a point on an elliptic curve. To do this we will use the addition operation on the curve:

- Definition: Suppose E is an elliptic curve defined over a field K, and P is a point on E. For any positive integer k, we define the point kP to be the sum $\underbrace{P + P + \cdots + P}_{k \text{ terms}}$, and we also define $(-k)P$ to be the additive inverse $-(kP)$ along with $0P = \infty$. The smallest positive k for which $kP = \infty$ is then called the order of P; if no such k exists, then we say P has infinite order. A point of finite order is called a torsion point and a point with $mP = \infty$ is called an m-torsion point.

[1] For completeness, however, we will outline a proof in this footnote. First, observe that the p-power Frobenius map $\varphi : E \to E$ defined via $(x, y) \mapsto (x^p, y^p)$ is a well-defined homomorphism from the group of points on E to itself (such a map is called an endomorphism of E) and has degree p. Then the group $E(\mathbb{F}_p)$ of $\mathbb{F}_p$-rational points is the kernel of $1 - \varphi$, so $\deg(1-\varphi) = \#E(\mathbb{F}_p)$ and the map $1 - \varphi$ can also be shown to be separable. Finally, applying the Cauchy-Schwarz inequality to the positive-definite quadratic form given by the degree map on the space of endomorphisms of E yields the result: one has $|\deg(1-\varphi) - \deg(\varphi) - \deg(1)| \le 2\sqrt{\deg(\varphi)\deg(1)}$, which reduces to $|\#E(\mathbb{F}_p) - p - 1| \le 2\sqrt{p}$ as claimed.

- Remark (for those who like group theory): This is the same as the usual definition of the order of an element of a group and the torsion elements of an abelian group.
- Compare this definition to the one in modular arithmetic: the order of a unit u modulo m is the smallest $k > 0$ such that $u^k \equiv 1 \pmod{m}$.
- Note that kP is well-defined because the addition law is associative: it does not matter the order in which we perform the additions. Likewise, we can see more or less immediately that $(a+b)P = aP + bP$ for any integers a and b.
- Over the real or complex numbers, "most" points on an elliptic curve will have infinite order: more precisely, the set of torsion points is countably infinite, while the set of all points is uncountable.
- As we will show, however, on an elliptic curve modulo p all points have finite order.

- Example: Find the order of the point $P = (1,3)$ on the elliptic curve $E : y^2 = x^3 + 4x + 4$ modulo 5.
  - We simply compute the multiples of P using the addition law repeatedly.
  - We obtain $2P = P + P = (2,0)$, $3P = 2P + P = (1,2)$, $4P = 3P + P = \infty$.
  - Since $4P$ is the smallest multiple of P that gives the point ∞, the order of P is $\boxed{4}$.

- We can compute large multiples of a particular point using successive doubling, in analogy to the procedure of successive squaring:

- Algorithm (Successive Doubling): To compute kP, first find the binary expansion of $k = \underline{b_j b_{j-1} \cdots b_0}$. Then compute the multiples $2P$, $4P$, $8P$, ... , $2^j P$ by using the doubling part of the addition law. Finally, compute $kP = \sum_{\substack{0 \le i \le j \\ b_i = 1}} 2^{b_i} P$ using the addition law.
  - We can speed this procedure up a bit by also using subtractions: unlike with modular arithmetic, where it is comparatively expensive to compute inverses, if $P = (x,y)$ then we have the trivial formula $-P = (x, -y)$.
  - We will also observe that this procedure works for any elliptic curve, not just an elliptic curve modulo p. The only issue is that large multiples of a typical point will usually grow very complicated over an infinite field.

- Orders of points on an elliptic curve share many of the same properties as orders of units modulo an integer m, and the proofs of these results are also essentially the same.

- Proposition (Properties of Order on Elliptic Curves): Suppose E is an elliptic curve and P is a point on E.
  1. If P has finite order k and $mP = \infty$, then k divides m.
     - Proof: Suppose $mP = \infty$ and write $m = qk + r$ where $0 \le r < k$.
     - We then have $rP = mP + (-qk)P = mP + (-q)(kP) = \infty + (-q)\infty = \infty + \infty = \infty$.
     - Since $rP = \infty$ and $0 \le r < k$, the only possibility is to have $r = 0$: otherwise this would contradict the minimality of k. Thus $m = qk$ so k divides m.
  2. If $mP = \infty$ but $(m/q)P \ne \infty$ for any prime divisor q of m, then P has order m.
     - Proof: Suppose the order of P is k. Then since $mP = \infty$, by (1) we conclude that k divides m.
     - If $k < m$, then there must be some prime q in the prime factorization of m that appears to a strictly lower power in the factorization of k: then k divides m/q.
     - But then $(m/q)P = \infty$ since m/q is a multiple of k, but this is contrary to the given information. Thus $m = k$ so P has order m.
  3. If E is an elliptic curve modulo a prime p and N is the number of points on E modulo p, then $NP = \infty$. In particular, the order of P divides N.
     - This result is an analogue of Euler's theorem. It is a corollary of a more general result of group theory known as Lagrange's theorem, which states that the order of any element of a group divides the number of elements in the group.

  - In our case, we can adapt the proof of Euler's theorem with minimal difficulty.
  - Proof: Suppose the points on E are $Q_1, Q_2, \cdots, Q_N$ and consider the points $Q_1+P, Q_2+P, \cdots, Q_N+P$: we claim that they are simply the points $Q_1, Q_2, \cdots, Q_N$ again (possibly in a different order).
  - Since there are N points listed and they all lie on the curve E, it is enough to verify that they are all distinct.
  - So suppose $Q_i+P = Q_j+P$. Then we can write $Q_i = Q_i+\infty = Q_i+(P+(-P)) = (Q_i+P)+(-P) = (Q_j+P)+(-P) = Q_j+(P+(-P)) = Q_j+\infty = Q_j$, where we used associativity and the properties of ∞ and inverses. (Morally, we simply subtracted P from both sides.)
  - Thus the points $Q_1+P, Q_2+P, \cdots, Q_N+P$ are simply $Q_1, Q_2, \cdots, Q_N$ in some order. Adding up all the terms then yields $(Q_1+P)+\cdots+(Q_N+P) = Q_1+\cdots+Q_N$, and upon rearranging and subtracting $Q_1+\cdots+Q_N$ from both sides (in the same way as above), we obtain $NP = \infty$ as desired.
  - The second statement follows immediately from $NP = \infty$ and (1) above.
  - Remark: The only fact we actually required here is that E had finitely many points. In general, if we can establish that E has finitely many points N (over any field K, not necessarily $\mathbb{F}_p$), then the order of any point on E divides N.

- Example: Show that the point $P = (1,3)$ has order 15 on the elliptic curve $E : y^2 = x^3 + 4x + 4$ modulo 13.
  - It is a straightforward check that $15P = \infty$ using successive doubling: we compute $2P = (12,8)$, $4P = (6,6)$, $8P = (0,11)$, $16P = (1,3)$. Then $15P = 16P - P = (1,3) - (1,3) = \infty$.
  - Furthermore, we can compute $3P = 2P + P = (3,2)$ and $5P = 4P + P = (10,2)$.
  - Since neither of these quantities is ∞, we conclude that the order of P must be $\boxed{15}$.

- If we can compute the orders of some points on E, we can often use that information in conjunction with the Hasse bound to determine the number of points on E without actually computing them all.
  - In the above example, we exhibited a point of order 15 on the elliptic curve $E : y^2 = x^3 + 4x + 4$ modulo 13. Thus, by our results on orders, the number of points on E must be a multiple of 15.
  - By the Hasse bound, the number of points on E must satisfy $|N - 14| \le 2\sqrt{13}$, yielding the inequality $6.78 \le N \le 21.22$. The only multiple of 15 in this range is 15 itself, so E must have exactly 15 points.

7.2 Factorization and Cryptography with Elliptic Curves

- Now that we have a reasonably good analogy between modular multiplication and the points on an elliptic curve modulo p under addition, we can use these analogies to develop algorithms for computational number theory and cryptography.
  - We will first discuss how to use elliptic curve arithmetic to design an integer factorization algorithm.
  - We then discuss how to develop several cryptographic protocols relying on the addition law on an elliptic curve. These will include a public-key cryptosystem based on ElGamal encryption, a key-exchange protocol based on Diffie-Hellman key exchange, and a digital signature algorithm.

7.2.1 Elliptic Curve Factorization

- We first explain how to create a factorization algorithm using elliptic curves based off of the method of Pollard's $(p-1)$-algorithm. These ideas were first proposed by H. Lenstra in 1985.
  - In Pollard's $(p-1)$-algorithm, the basic idea is that if $n = pq$ and we choose a random integer a, then the order of a modulo p is likely to differ from the order of a modulo q.
  - Thus, if the order of a mod p is k and is larger than k mod q, $a^k \equiv 1 \pmod{p}$ but $a^k \not\equiv 1 \pmod{q}$, so that $\gcd(a^k - 1, n) = p$.

- Let us now try to construct an appropriate analogy with elliptic curves:

- Again, suppose $n = pq$ is a product of two primes, and suppose we choose a (nonsingular) elliptic curve $E : y^2 = x^3 + Ax + B$ over the integers along with a point P on the curve.
- The order of P on E_p, the reduction of E modulo p, is unlikely to be exactly equal to the order of P on E_q, the reduction of E modulo q.
- If the order of P on E_p is k and the order of P on E_q is larger than k, then $kP = \infty$ on E_p but $kP \neq \infty$ on E_q.
- Now the question arises: how can we detect this behavior? In Pollard's $(p-1)$-algorithm, we performed all our calculations modulo n, so it seems we should do the same thing here.
- Thus, we do all of our computations on the curve E_n, the reduction of the curve E modulo n, using the addition law formulas defined over the rational numbers reduced modulo n.
- Assuming that this reduction is well-defined, the addition law will still obey all of the requirements we put on it (namely, it will be commutative, associative, have an identity ∞, and have inverses).
- However, the addition law formulas require a division when computing the slope of the line, and if this slope requires dividing by a nonzero number that is not invertible mod n, then we will not be able to evaluate the result. (If we were dividing by zero itself, then we would simply obtain a slope of ∞: however, there is no sensible way to interpret a slope of $\frac{1}{2}$ modulo 6.)
- This is precisely what we want: it is saying that the slope of the line is ∞ modulo one of the prime divisors of n, but not ∞ modulo the other. Then to find the nontrivial divisor of n, we simply take the gcd of the problematic denominator with n.
- Another way to interpret this idea is using the Chinese remainder theorem: a point (x, y) lies on E_n if and only if it lies on the curve $E_p : y^2 = x^3 + Ax + B$ modulo p and the curve $E_q : y^2 = x^3 + Ax + B$ modulo q.
- Thus, the points on E_n can equivalently be thought of as pairs of points (P, Q) of points on E_p and E_q. We are then seeking to detect when a multiple of a pair (P, Q) is ∞ in one coordinate but not in the other.

• <u>Example</u>: Examine what happens when trying to add the point $P_1 = (1, 3)$ to the point $P_2 = (15, 4)$ on the elliptic curve $E_{21} : y^2 = x^3 + 4x + 4$ modulo 21, and when doubling the point P_1.

- To find $P_1 + P_2$ we compute the slope of the line: it is $\frac{4-3}{15-1} = \frac{1}{14}$. However, this quotient is not defined modulo 21, since 14 is not relatively prime to 21. In this case, we see that $\gcd(21, 14) = 7$ is a proper divisor of 21.
- Similarly, if we try to compute $P_1 + P_1$, the slope of the tangent line is $\frac{3(1)^2 + 4}{2 \cdot 3} = \frac{7}{6}$, which is again not defined modulo 21 since 6 is not relatively prime to 21. In this case, we see that $\gcd(21, 6) = 3$ is a proper divisor of 21.
- Ultimately, what is happening in the first case is that $P_1 + P_2 = \infty \pmod 7$ but $P_1 + P_2 \neq \infty \pmod 3$. In the second case, $2P_1 = \infty \pmod 3$ but $2P_1 \neq \infty \pmod 7$.

• To implement this procedure to factor integers in a reasonable way requires a bit more care, but again we can take guidance from Pollard's $(p-1)$-algorithm.

- Searching through all possible k in Pollard's $(p-1)$-algorithm is very inefficient. To speed things up, we observe that it is unnecessary to find the exact order of a mod p: any multiple of it will suffice, as long as that multiple is not also divisible by the order of a mod q.
- A reasonably efficient procedure is to evaluate $\gcd(a^{d!} - 1, n)$ for $1 \leq d \leq M$ (for some choice of bound M) until we obtain a gcd that is larger than 1.
- In the elliptic curve analogy, we should therefore try computing $(d!)P$ on an elliptic curve $E_n : y^2 = x^3 + Ax + B$ modulo n for $1 \leq d \leq M$, and seeing if we obtain a denominator that has a nontrivial gcd with n in the denominator. If we do, then we get a factorization of n.
- The only remaining question is how to choose an elliptic curve E along with a point P. An easy way to generate a pair (E, P) is to choose the coordinates of $P = (x_0, y_0)$ along with the value A first, and then set $B = y_0^2 - x_0^3 - Ax_0$.

- Lenstra's algorithm is simply a reformulation of these ideas:
- Algorithm (Lenstra's Factorization Algorithm): Suppose n is composite. Choose a bound M, a point $P = (x_0, y_0)$, and an integer A. Let E_n be the elliptic curve $y^2 = x^3 + Ax + B$ modulo n, with B chosen so that P lies on E_n. Set $Q_1 = P$ and for $2 \le j \le M$, define $Q_j = jQ_{j-1}$ (computed on E_n). If at any stage of the computation the point Q_j cannot be computed, due to a necessary division by a denominator d which is not 0 modulo n but which is not invertible modulo n, then $\gcd(d, n)$ is a proper divisor of n. If a divisor is not found and Q_M is not ∞, increase the value of M and continue the computation. Otherwise, if $Q_M = \infty$, repeat the procedure with a new choice of P and A.
  - We will remark that the curve E can be singular, as long as P is not the singular point on the curve. (By "singular" we mean singular mod p or mod q, which is equivalent to saying that the discriminant Δ has a common prime divisor with n.)
  - However, choosing E to be a singular curve is not optimal, because (as it turns out) the algorithm will essentially reduce either to Pollard's $(p-1)$-algorithm or trial division according to the type of singularity.
- Example: Use Lenstra's factorization algorithm to find a divisor of the integer $n = 170999$ using the point $P = (1, 4)$ on the elliptic curve $E : y^2 = x^3 + 4x + 11$.
  - We simply compute the points Q_j successively using the recursion $Q_1 = P$, $Q_j = jQ_{j-1}$ on the elliptic curve E modulo n until we obtain a problematic denominator.

j	1	2	3	4	5
Q_j	$(1, 4)$	$(109545, 75144)$	$(81282, 86818)$	$(100818, 143145)$	$(152033, 116998)$
Factor?	no	no	no	no	no
j	6	7	8	9	10
Q_j	$(87978, 17295)$	$(104368, 99929)$	$(126411, 167685)$	$(79623, 108587)$	–
Factor?	no	no	no	no	557

  - In this case, attempting to compute $10Q_9$ will require dividing by a denominator that is not relatively prime to n.
  - The exact details of the computation will depend on the method used to compute $10Q_9$, but successive doubling will yield $2Q_9 = (147257, 97701)$ and $8Q_9 = (160625, 116187)$, and attempting to add these two points will require using a line with slope $m = \dfrac{116187 - 97701}{160625 - 147257} = \dfrac{18486}{13368}$, and $\gcd(13368, 170999) = 557$.
- The elliptic curve factorization algorithm seems to work, but it is not obvious how fast it is nor how efficient it is in comparison to our other algorithms.
  - As we noted above, the factorization algorithm will succeed after M steps when the order of P on the elliptic curve E_p (i.e., E modulo p) divides $M!$, but the order of P on E_q (i.e., E modulo q) does not divide $M!$.
  - It is unlikely that these two things will occur at exactly the same value of M, so what we are really seeking is for the order of P on E_p to divide $M!$.
  - From our results on orders, we know that the order of P on E_p divides the number of points N on E_p, so we are certainly guaranteed to succeed if N divides $M!$.
  - Furthermore, by the Hasse bound, $|N - p - 1| \le 2\sqrt{p}$. It is in fact known that N can take any integral value in this interval, and (conjecturally) it does so according to a distribution that is not far from being uniform.
  - Thus, the elliptic curve factorization will succeed quickly as long as the prime divisors of N are all fairly small.
  - Note that this is a similar criterion to that of Pollard's $(p-1)$-algorithm, which succeeds quickly as long as the prime divisors of $p - 1$ are all fairly small. (An integer all of whose prime divisors are $\le M$ is called M-smooth.)
  - However, we are free to make different choices for the elliptic curve E, each of which will give a different random integer that is near p. As long as one of the curves we choose is M-smooth, we will obtain the factorization of n.

- Thus, elliptic curve factorization is much more versatile than Pollard's $(p-1)$-algorithm, because in the latter if $p-1$ has a large prime divisor then we are simply out of luck, whereas with elliptic curve factorization if N has a large prime divisor then we can simply switch to a different curve. (Of course, we will generally not know the exact value of N, so we would instead switch curves if we have spent a long time computing and not gotten any results yet.)

- Another advantage to using several curves is that the computations can be completely parallelized (i.e., they can be run on separate processors), since the point operations on different curves have nothing to do with one another.
  - It is a rather nontrivial analytic number theory problem to determine the appropriate heuristic for the density of integers in the "Hasse interval" $|N-p| \leq 2\sqrt{p}$ that are M-smooth, which is needed in order to estimate how many curves should be used in order to search for the factorization and to estimate the value of M that should be used.
  - We will not give the details of this computation, but the approximately optimal pairs (M, k) for the bound M and the number of curves k are roughly $(2000, 25)$ for 15-digit prime divisors, $(10000, 100)$ for 20-digit prime divisors, and $(50000, 300)$ for 25-digit prime divisors.
  - Overall, if one computes the total time requirement with optimal choices for the parameters, Lenstra's elliptic curve algorithm can factor an integer n in a total of approximately $e^{\sqrt{2}(\ln p)^{1/2}(\ln \ln p)^{1/2}}$ steps, where p is the smallest prime divisor of n.
  - This number of steps is bounded above by $e^{(\ln n)^{1/2}(\ln \ln n)^{1/2}}$, and so the elliptic curve factorization has roughly the same asymptotic speed as the quadratic sieve.
  - In practice, due to the fact that elliptic curve operations are slower than modular exponentiations, Lenstra's algorithm becomes slower than the sieve methods for integers exceeding 60 digits or so, and is slower than Pollard's ρ-algorithm for numbers under 30 digits.
  - However, the elliptic curve method is much more efficient at finding comparatively small divisors (around 30 digits or less) of large integers than the sieve methods are.
  - To factor a large integer that is not expected to be the product of only large primes (e.g., an RSA modulus), one often uses some combination of trial division, the Pollard $(p-1)$ algorithm, and the Pollard ρ algorithm to search for small factors (under 15 digits or so) and Lenstra's algorithm to search for factors of medium size (15-30 digits). Then one uses a sieve method to factor the remaining integer, which will be a product only of large primes.

- Finally, we will mention that there are several improvements and optimizations that can be made to Lenstra's original algorithm.
  - The largest computational overhead in Lenstra's algorithm is computing the point multiplications. There are various ways to arrange the arithmetic operations in such a way that fewer computations are needed: in particular, it is possible to use both additions and subtractions when doing successive doubling (since computing the inverse of a point is essentially free). Furthermore, by using different models for elliptic curves other than the reduced Weierstrass form $y^2 = x^3 + Ax + B$, further savings are possible.
  - It is also possible to choose the elliptic curve $y^2 = x^3 + Ax + B$ in such a way that it is still essentially random modulo n, but is guaranteed to have points of various small orders (such as 12). Such restrictions would then necessarily imply that the number of points on the curve is divisible by 12, marginally reducing the size of potential large prime divisors of N.
  - There are also "second stage" methods for Lenstra's algorithm (initially proposed by R. Brent[2]) that apply a procedure similar to Pollard's ρ-algorithm to try to find a factorization.
  - Briefly, if we compute $Q = (M!)P$ where M is fairly large but do not obtain a factorization, then (barring anything particularly weird) the order of Q on E_p will not have any prime divisors less than M.
  - Suppose that Q ends up having prime order on E_p. If we can find two "random" multiples of $k_1 Q = (x_1, y_1)$ and $k_2 Q = (x_2, y_2)$ that are equal on E_p but not on E_q, then $\gcd(y_2 - y_1, n)$ will be equal to p.

[2] See the paper "Some Integer Factorization Algorithms using Elliptic Curves" by R.P. Brent, Australian Comp. Sci. Comm. 8 (1986), available on arXiv:1004.3366.

- It would not be efficient to search directly for such multiples: instead, we could generate k such points using a "random function" (e.g., one that doubles a point half the time, and doubles then adds Q the other half of the time). Then we would compute the product $d = \prod_{1 \le i < j \le k}(y_i - y_j)$, evaluate $\gcd(d, n)$, and hope that it gives p.
- As with the analysis of Pollard's ρ-algorithm, we would want to take $k \approx 2\sqrt{r}$ where r is the expected order of Q. Efficiently evaluating the product modulo n is a rather nontrivial task (since it contains about $k^2/2$ terms), but there are methods for doing this, and overall it is possible to glean some small time savings over Lenstra's original algorithm.

7.2.2 Encoding Messages for Elliptic Curve Cryptography

- In order to use elliptic curves for cryptography, we must first encode messages as points on elliptic curves.
- Unlike with cryptosystems based on modular arithmetic, where we can simply write a message as a residue class modulo m (possibly with some kind of padding scheme to increase security), it is not quite so trivial to encode a message as a point on an elliptic curve if we specify the curve E ahead of time, as would be necessary for a public-key cryptosystem.
  - So suppose we have chosen an elliptic curve $y^2 = x^3 + Ax + B$ modulo a prime p, and wish to convert a message m into a point on the curve. We can assume that m is smaller than p, since we may break m up into pieces and send each piece separately using whatever scheme we come up with.
  - However we cannot, for example, simply convert a message m into the point (m, y) on the elliptic curve, because there may not be a value of y satisfying the equation $y^2 = m^3 + Am + B \pmod{p}$.
- To go further, we need to recall some results about quadratic residues modulo p.
- Definition: If a is a residue class modulo p, we say a is a quadratic residue if there is some b such that $b^2 \equiv a \pmod{p}$. If there is no such b, then we say a is a quadratic nonresidue.
  - It is straightforward to list the quadratic residues by squaring all of the residue classes.
  - Example: Modulo 5, the quadratic residues are 0, 1, and 4, while the nonresidues are 2 and 3.
  - Example: Modulo 7, the quadratic residues are 0, 1, 4, and 2, while the nonresidues are 3, 5, and 6.
  - Example: Modulo 13, the quadratic residues are 0, 1, 4, 9, 3, 12, and 10, while the nonresidues are 2, 5, 6, 7, 8, and 11.
  - In general, there are $(p+1)/2$ quadratic residues modulo p; namely, the values 0^2, 1^2, 2^2, ... , $((p-1)/2)^2$.
- The Legendre symbol is a useful tool that detects squares modulo p:
- Definition: If p is an odd prime, the Legendre symbol $\left(\frac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue, -1 if a is a quadratic nonresidue, and 0 if $p|a$.
  - The notation for the Legendre symbol is somewhat unfortunate, since it is the same as that for a fraction inside parentheses; it is nonetheless standard. When appropriate, we may write $\left(\frac{a}{p}\right)_L$ to emphasize that we are referring to a Legendre symbol rather than a fraction.
  - Example: We have $\left(\frac{2}{7}\right) = +1$, $\left(\frac{3}{7}\right) = -1$, and $\left(\frac{0}{7}\right) = 0$, since 2 is a quadratic residue and 3 is a nonresidue modulo 7.
  - Example: We have $\left(\frac{3}{13}\right) = \left(\frac{-3}{13}\right) = +1$, and $\left(\frac{2}{15}\right) = 1$, since 3 and -3 are quadratic residues modulo 13, while 2 is not.
  - Note that the quadratic equation $x^2 \equiv a \pmod{p}$ has exactly $1 + \left(\frac{a}{p}\right)$ solutions modulo p.

- In general, if we have a primitive root u modulo p, then a unit a is a quadratic residue if and only if it is an even power of u: if $a = u^{2k}$ then $(u^k)^2 = a$, and conversely if $a = b^2$ then $b = u^k$ is some power of u, and then $a = u^{2k}$ is an even power of u.
- Using this observation we can give a much faster method for computing the Legendre symbol:

- Theorem (Euler's Criterion): If p is an odd prime, then for any residue class a, it is true that $\left(\frac{a}{p}\right) = a^{(p-1)/2}$ (mod p).
  - Proof: If $p|a$ then both sides are zero, so now assume a is a unit modulo p and let u be a primitive root modulo p.
  - First suppose a is a quadratic residue, so that $\left(\frac{a}{p}\right) = +1$. By the proposition above, we know $a = u^{2k}$ for some integer k; then $a^{(p-1)/2} \equiv (u^{2k})^{(p-1)/2} = (u^{p-1})^k \equiv 1^k = 1 \pmod{p}$, which agrees with $\left(\frac{a}{p}\right)$.
  - Now suppose a is a quadratic nonresidue, so that $\left(\frac{a}{p}\right) = -1$. Again by the proposition above, we know $a = u^{2k+1}$ for some integer k; then we compute $a^{(p-1)/2} \equiv (u^{2k+1})^{(p-1)/2} = (u^{p-1})^k \cdot u^{(p-1)/2} \equiv u^{(p-1)/2}$.
  - Now observe that $x = u^{(p-1)/2}$ has the property that $x^2 \equiv 1 \pmod{p}$. The two solutions to this quadratic are $x \equiv \pm 1 \pmod{p}$, but $x \not\equiv 1 \pmod{p}$ since otherwise u would not be a primitive root.
  - Hence $u^{(p-1)/2} \equiv -1 \pmod{p}$, meaning that $a^{(p-1)/2} \equiv -1 \pmod{p}$ as well, and this agrees with $\left(\frac{a}{p}\right)$.

- Example: Determine whether $a = 17441$ and $b = 135690$ are quadratic residues modulo the prime $p = 239441$.
  - We simply compute $a^{(p-1)/2} \equiv a^{119720} \equiv 1 \pmod{p}$, so by Euler's criterion $\boxed{a \text{ is a quadratic residue}}$ mod p.
  - Likewise, $b^{(p-1)/2} \equiv b^{119720} \equiv -1 \pmod{p}$, so by Euler's criterion $\boxed{b \text{ is not a quadratic residue}}$ mod p.

- As one of many corollaries of Euler's criterion, we can deduce that the Legendre symbol is multiplicative:

- Corollary: For any odd prime p, the Legendre symbol modulo p is multiplicative: $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. In particular, the product of two quadratic nonresidues is a quadratic residue.
  - Proof: Observe $\left(\frac{ab}{p}\right) \equiv (ab)^{(p-1)/2} \equiv a^{(p-1)/2} b^{(p-1)/2} \equiv \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \pmod{p}$.

- Finally, we will recall a useful result that allows us to compute square roots modulo a prime congruent to 3 modulo 4:

- Proposition: If p is a prime congruent to 3 modulo 4 and a is a quadratic residue modulo p, then $x = a^{(p+1)/4}$ has $x^2 \equiv a \pmod{p}$.
  - Proof: Since $a = m^2 \pmod{p}$ by hypothesis and $m^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem, we can then write $x^2 \equiv a^{(p+1)/2} \equiv m^{p+1} \equiv m^2 \equiv a \pmod{p}$.
  - We will also remark that there are fast algorithms for computing square roots of quadratic residues modulo primes congruent to 1 modulo 4, but they are more complicated.

- We can now return to the question of encoding messages on an elliptic curve $E : y^2 = x^3 + Ax + B$ modulo p, where we will now also take $p \equiv 3 \pmod{4}$.
  - From the above analysis, we would expect, based on the fact that $(p+1)/2$ of the residues modulo p are squares, that for any given x there should exist a y with $y^2 = x^3 + Ax + B$ mod p about half of the time.
  - If we try to encode a message directly as the x-coordinate of a point, we therefore should only expect to succeed about half of the time.

- A better procedure is instead to encode a message as part of the x-coordinate of a point, and then try to choose the remaining piece of the x-coordinate in such a way that $x^3 + Ax + B$ is a quadratic residue modulo p.
- Here is a particular scheme for doing this: if p has $r + k + 1$ bits when written in base 2, we break the message into pieces each containing r bits.
- Then, to convert an r-bit message m, we pad the beginning m with $k+1$ bits: a zero followed by k bits $b_1 b_2 \cdots b_k$ that can be arbitrarily chosen, and set x to be the bit string $0b_1 \cdots b_k m$.
- We then search through the possible choices of these k bits until we find a solution y to $y^2 = x^3 + Ax + B \pmod p$, and pick one of the two possible values of y arbitrarily. We then perform our encryption procedure using the point (x, y) on E modulo p.
- To recover the message m from a point (x, y), where $0 \le x < p$ we simply compute x modulo 2^r and write the result as a bit string in base 2.
- Ultimately, since there are 2^k possible choices for the bit string $b_1 b_2 \cdots b_k$, the probability that none of them yields a quadratic residue $x^3 + Ax + B$ is roughly $1 - 2^{-2^k}$. (Of course, the probabilities are not entirely independent, but they should be nearly so.)
- Even if we merely take $k = 10$, this probability is already so vanishingly small that it is unlikely a problem would ever occur in practical deployment.

- <u>Example</u>: Encode the message $m = 13 = 1101_2$ as a point on the elliptic curve $y^2 = x^3 + 11x + 17$ modulo $p = 307$ using a message length $r = 4$ bits and a padding length of $k = 4$ bits.
  - We note that $p > 256 = 2^8$ so p has 9 bits in base 2.
  - We therefore want to search for a bit string $b_1 b_2 b_3 b_4$ such that $x = 0b_1 b_2 b_3 b_4 1101_2$ is a quadratic residue modulo 307.
  - The bit string 0000 yields the value $x = 13$, but $x^3 + 11x + 17 \equiv 208 \pmod{307}$ is a quadratic nonresidue as can be confirmed by evaluating $208^{153} \equiv -1 \pmod{307}$.
  - The bit string 0001, however, yields $x = 29$, and $x^3 + 11x + 1 \equiv 165 \pmod{307}$ is a quadratic residue as can be confirmed by evaluating $165^{153} \equiv 1 \pmod{307}$.
  - To compute the associated value of y, we then compute $x^{(p+1)/4} \equiv 29^{77} \equiv 120 \pmod{307}$. A point associated to m on the curve E is then $\boxed{(29, 120)}$.
  - Of course, there are many other such points: another is the additive inverse $(29, 187)$.
  - We could also have searched more randomly for possible bit strings (rather than starting at 0000 and going upward), to try to keep the procedure from being as predictable. The bit string 1110, for example, yields another possible point $(237, 209)$.
  - To recover the message m, we simply extract the x-coordinate and reduce it modulo $2^4 = 16$. This yields the correct original message $13 = 1101_2$.

7.2.3 Public-Key Encryption with Elliptic Curves

- We now discuss the creation of public-key cryptosystems using elliptic curves, which was first proposed by Neal Koblitz and Victor Miller in 1985.
  - Based on our earlier discussion on how to convert messages to points on curves, we will assume throughout that our plaintext is a point (x, y) on a given elliptic curve E.
  - We will generally work with the reduction E_p of E modulo a prime p, and N will denote the number of points on E_p.
- A natural first guess for how to create a public-key cryptosystem would be to adapt RSA or Rabin encryption to the elliptic curve setting: however, some difficulties will arise if we try to do this.
  - An RSA/Rabin-like procedure would involve roughly the following: Bob creates a public key consisting of an elliptic curve E, a prime p, and an "encryption multiplier" e.

- If Alice wants to encrypt a plaintext message $P = (x, y)$, she computes the ciphertext $C = eP$ on E_p and sends it to Bob.
- To decrypt a ciphertext C, Bob then computes $P = dC$ for an appropriate "decryption" multiplier d.
- In order for everything to work properly, Bob needs $(de - 1)P = \infty$ for every possible message P. From our results on orders, this is essentially equivalent to requiring that $de \equiv 1 \pmod{N}$ where N is the number of points on E_p.
- Thus, to compute appropriate values of d and e, Bob would need to compute the number of points on E_p.
- As we have seen, this task is not entirely trivial, although there is a procedure known as <u>Schoof's algorithm</u> can compute the number of points on an elliptic curve modulo p in time approximately equal to $(\log p)^5$. (An improvement due to Elkies and Atkin can heuristically improve this result to $(\log p)^4$.)
- Roughly speaking, the idea of Schoof's algorithm is to compute the value of N modulo enough small primes that we can find N modulo r for a value of r larger than $4\sqrt{p}$: then the Hasse bound will yield a unique possible value of N.
- However, an immediate problem arises: in order for Eve to break the cryptosystem, it is clearly sufficient for her to compute the number of points on E_p, as she can then compute d the same way Bob does. If Eve has a computer that is at least as powerful as Bob's, then she can break the cryptosystem completely.
- Ultimately, there does not seem to be a good way to avoid this problem.
- Suppose we instead try to work with an elliptic curve modulo a nonprime integer $n = pq$: then the addition law will not always work properly. If we ignore that particular issue, the system is essentially using a pair of points (P, Q), one on E_p and one on E_q, and an appropriate pair (e, d) can be found as a solution to the congruence $de \equiv 1 \pmod{N_p N_q}$.
- However, in this case, Eve would be able to break the system by factoring n, since she could then compute the values N_p and N_q using Schoof's algorithm above. The usage of elliptic curves here does not add to the security, and merely serves to complicate everything.

• Instead of trying to use the difficulty of inverting modular exponentiation (which is only hard when the modulus is composite), we should instead try to build systems that rely on the difficulty of computing discrete logarithms, which is a more natural problem for elliptic curves modulo a prime p. We will therefore describe a procedure for an elliptic curve version of ElGamal encryption.

• First, Bob must create his public key.

- To do this, he chooses an elliptic curve E, a prime p, and a point Q_a on E whose order is large.
- Ideally, Bob should choose the point Q_a to have an order whose value is a large prime roughly equal to the number of points on the curve E_p, but this can be a bit hard to arrange.
- In our description of ElGamal encryption with modular arithmetic, Bob chose a value a which was a primitive root modulo p. It was not actually necessary to choose a primitive root: the system works essentially as well when a is any value whose order is sufficiently large that computing discrete logarithms to the base a is difficult.
- Bob can search for such a Q_a by computing $(M!)Q_a$ for a reasonably large value of M and making sure that it is not equal to ∞.
- Alternatively, Bob could try to find a curve E having a prime number of points on it: then any point other than ∞ will have order N.
- Bob then chooses a positive integer d that is less than the number of points on E_p (he does not actually need to compute the number of points itself, since he can just choose d to be less than $p - 2\sqrt{p}$) and computes the point $Q_b = dQ_a$.
- Bob then publishes (E, p, Q_a, Q_b), which serve as his public key.

• Now suppose that Alice wants to send Bob a message $P = (x, y)$.

- Alice chooses a random integer k less than the number of points on E_p (again, she could simply choose a random integer less than $p - 2\sqrt{p}$) and computes $Q_r = kQ_a$ and $Q_s = kQ_b + P$ on E_p.

- She then sends the pair (Q_r, Q_s) to Bob.

- If Bob has received a ciphertext pair (Q_r, Q_s), he wishes to recover the value of m.
  - To do this, Bob simply computes $Q_s - dQ_r = (kQ_b + P) - d(kQ_a) = P + kdQ_a - dkQ_a = P$ on E_p.
  - Note of course that Bob would compute the subtraction as $Q_s + d(-Q_r)$, where $-Q_r$ is the additive inverse of Q_r.

- Example: If Bob uses elliptic-curve ElGamal with $E : y^2 = x^3 + 7x + 1$, $p = 44927$, $Q_a = (7772, 14369)$, and $d = 22105$, find Bob's public key, encode the message $P = (14605, 29833)$, and then decode the associated ciphertext.
  - First, Bob computes $Q_b = dQ_a = (39061, 4109)$ using successive doubling. His public key then consists of the quadruple (E, p, Q_a, Q_b).
  - Now, if Alice wants to encode the message P, she chooses a random integer k less than $p - 2\sqrt{p} \approx 44503.08$. Imagine she chooses $k = 23207$.
  - She then computes $Q_r = kQ_a = (30566, 37885)$ and $Q_s = kQ_b + P = (35487, 8262) + P = (40194, 40273)$ and sends them to Bob.
  - Bob receives the ciphertext pair Q_r, Q_s, and then decrypts by evaluating $Q_s - dQ_r = Q_s + (35487, 36665) = (14605, 29833)$, which is indeed the correct plaintext.
  - Remark: For the given parameters, the curve E_p turns out to have a prime number of points (44651) so P necessarily has order 44651 on this curve.

- Like with cryptosystems based on modular arithmetic, the only steps required to implement elliptic curve ElGamal are the point operations on the elliptic curve, which can be done comparatively fast using the successive doubling algorithm. However, it is less obvious why the procedure is secure.
  - Suppose Eve intercepts the transmitted information: she will obtain (E, p) along with Q_a, Q_b, Q_r, and Q_s. She wants to compute $P = Q_s - kQ_b = Q_s - dkQ_a = Q_s - dQ_r$ on E_p.
  - If Eve knows d then she can decrypt using the same procedure Bob uses. However, in order to find d from Bob's public key, Eve would need to determine the value d for which $dQ_a = Q_b$, which is the elliptic curve analogue of computing a discrete logarithm.
  - Furthermore, since Alice chooses k randomly, $Q_r = kQ_a$ will essentially be a random point on the curve E_p (technically, it will be a random multiple of Q_a, but this does not tell Eve very much if Q_a has a large order). Likewise, $Q_s = kQ_b + P$ will be essentially random.
  - Knowing Q_r alone does not help, because in order to compute k Eve would again need to compute an elliptic-curve discrete logarithm. Knowing Q_s does not help much either, because in order for Eve to compute P she would have to know the value of kQ_b, which in turn would require knowing the value of k.
  - Ultimately, like with the modular version of ElGamal, the only obvious method of attack is to compute a discrete logarithm.

- It appears to be much harder to compute elliptic curve discrete logarithms than modular discrete logarithms. Several of the simpler systems have natural analogues:
  - There is a version of the Pohlig-Hellman algorithm that will be effective when the number of points N on E_p has only small prime divisors. (In this case, N plays the role of $p - 1$ in the algorithm.) This situation is easy to avoid if the curve E is chosen properly.
  - There is also a version of the baby-step giant-step method whose procedure is essentially identical and requires approximately $p^{1/2}$ steps to compute a discrete logarithm.
  - Here is the algorithm, for completeness: to find a solution to $dQ_a = Q_b$ on an elliptic curve E_p modulo p, choose an integer M such that $M^2 \geq N$, where N is the number of points on E_p. Compute two lists: the points xQ_a for all $0 \leq x \leq M - 1$ and the points $Q_b - MyQ_a$ for all $0 \leq y \leq M - 1$. Then compare the two lists to find an element that is on both lists: if $xQ_a = Q_b - MyQ_a$, we get a solution $d = x + My$.

- However, there does not appear to be any natural analogue of any of the sieving algorithms.
  - The basic reason is that the sieving algorithms all rely on an easily-computed notion of "smallness" of a residue class modulo n that remains consistent under modular multiplication (i.e., the product of two small numbers modulo n remains small modulo n). The idea is then to try to obtain a large number of relations among small primes and use them to compute the discrete logarithms of enough small primes to allow new discrete logarithms to be computed rapidly.
  - However, there is no analogous notion of size that is easy to compute on an elliptic curve modulo p: for one thing, even if the x-coordinate of a point is small, the y-coordinate will look more or less random and very often will be large.
  - Also, even if all the coordinates of particular points are both small, their sum may have very large coordinates due to the modular divisions in the addition law.
  - Finally, even if we were to declare that a point is "small" if it had a small x-coordinate, there is no easy way to see how a large point can be written as a sum of small points that is analogous to the way we can easily factor a big integer that is a product of small primes.
- Since the sieving algorithms do not carry over, and there do not seem to be any other natural algorithms that are comparable, we can achieve a level of security comparable to that of RSA using an elliptic curve cryptosystem with much smaller key sizes.
  - It is estimated, based on the speed of integer factorization algorithms versus the speed of elliptic curve discrete logarithm algorithms, that an elliptic curve cryptosystem with a key size of 256 bits provides security roughly comparable to that of RSA with a key size of approximately 3000 bits.
  - The smaller key size leads to significant savings in computation time, even after accounting for the additional complexity of doing elliptic curve addition versus modular multiplication.
  - In actual practice, since it is a nontrivial problem to count the number of points on a given elliptic curve, many elliptic curve protocols specify using a curve published by an independent authority, such as NIST, that has done the point-counting ahead of time and certifies it as being secure. Of course, this requires a degree of trust that the authority has not intentionally chosen a curve that has some kind of nonobvious "backdoor" (i.e., some clever way of computing discrete logarithms quickly), though in practice it seems unlikely such a backdoor would exist.
  - Many implementations use elliptic curves defined over finite fields of characteristic 2, since such fields are particularly amenable to binary arithmetic. Such fields will have 2^n elements for some integer n and have a structure similar to the integers modulo 2^n, except with a different type of multiplication that makes all of the nonzero elements into units.
  - We will not delve into the rich and interesting area of finite fields, but we will reiterate that it is necessary to work with a different Weierstrass model over a field of characteristic 2.

7.2.4 Key Exchange and Digital Signatures with Elliptic Curves

- Next, we describe how to create a version of Diffie-Hellman key exchange for elliptic curves using the same techniques.
- First, Alice and Bob jointly choose a large prime p, an elliptic curve E_p modulo p, and a point P on E having large order.
  - Again, as with the construction for ElGamal encryption, there are moderately straightforward procedures for generating this triple (E, p, P).
  - One option is to choose p and then search for an elliptic curve E_p whose number of points is a prime or a prime times another small number d (e.g., 4). Then for any point P, if $dP \neq \infty$, it follows that P must have large order.
  - It is also possible to use a curve certified by a trusted authority that has done the point-counting already, although this involves taking the risk that Eve has precomputed a lot of information about that curve for the purposes of trying to break cryptographic protocols that use it.

- Alice then chooses a secret positive integer a less than the order of P and sends Bob the point $Q_a = aP$ on E_p.
- Likewise, Bob chooses a secret positive integer b less than the order of P and sends Alice the point $Q_b = bP$ on E_p.
- Their secret shared key is then the point $Q_{ab} = abP = a(Q_b) = b(Q_a)$, which can be computed by both Alice and Bob using their secret number along with the point shared by the other.

- Example: Use elliptic-curve Diffie-Hellman to construct a secret shared key using $E : y^2 = x^3 + 7x + 1$, $p = 44927$, and $P = (27844, 29401)$, where Alice's secret number is $a = 40006$ and Bob's secret number is $b = 18846$.
  - Alice computes $Q_a = aP = (3454, 34367)$ and sends it to Bob. Bob computes $Q_b = bP = (22472, 6971)$ and sends it to Alice.
  - Alice then recovers $Q_{ab} = aP_b = (2147, 22480)$ and Bob recovers $Q_{ab} = bQ_a = (2147, 22480)$.
  - Bob and Alice now have a secret shared key $Q_{ab} = (2147, 22480)$ that they can use for further communications (e.g., with a symmetric-key cryptosystem).

- If Eve is eavesdropping on the conversation, she will know E_p along with P, Q_a, and Q_b , and she wants to compute Q_{ab}.
  - In order to do this, Eve would essentially need to compute one of the multipliers a and b. Since P is assumed to have large order, the only reasonable way to do this is for her to evaluate a discrete logarithm on E_p.
  - Again, as we have already discussed, computation of discrete logarithms on elliptic curves appears to be very difficult.
  - It is of course possible that there is some way to combine the information in P, Q_a, Q_b to find Q_{ab}, but this seems unlikely since the operations of scaling a point by a and scaling a point by b are essentially independent.

- Like with modular Diffie-Hellman, we can extend this basic protocol to include more than two participants.
  - For example, if Alice, Bob, and Carol wish to construct a secret shared key together, they collectively agree on a triple (E, p, P) and choose their own secret numbers a, b, and c each less than the order of P on E_p.
  - Alice then publishes $Q_a = aP$, Bob publishes $Q_b = bP$, and Carol publishes $Q_c = cP$.
  - Next, Alice computes $Q_{ab} = aQ_b$ and publishes it, Bob computes $Q_{bc} = bQ_c$ and publishes it, and Carol computes $Q_{ac} = cQ_a$ and publishes it.
  - Each person then computes the shared secret key $Q_{abc} = aQ_{bc} = bQ_{ac} = cQ_{ab}$ using their secret and the public information.
  - If Eve is eavesdropping, she will have P, aP, bP, cP, abP, acP, bcP, but not the secret $abcP$. There is no obvious way she can compute the secret that does not essentially require computing a discrete logarithm somewhere.

- Also, like with the basic implementation of modular Diffie-Hellman, this protocol does not have any authentication and is therefore susceptible to the "man-in-the-middle" attack wherein Mallory impersonates Alice to Bob and simultaneously impersonates Bob to Alice, and performs a simultaneous key exchange with both of them.
  - One way to include an authentication step would be for both of Alice and Bob to put a digital signature on their communications during the key creation process, so that the other person feels confident that Mallory is not impersonating either of them.

- We will now describe how to adapt the ElGamal signature algorithm to the elliptic curve setting. Some details of the algorithm differ from the modular case since we are dealing with points rather than individual numbers.

- Alice first creates an elliptic-curve ElGamal public key (p, E, Q_a, Q_b) where p is a large prime, E is an elliptic curve modulo p on which it is hard to compute discrete logarithms, Q_a is a point on E whose order has only large prime factors, and $Q_b = dQ_a$ for Alice's secret number d.
- Alice also calculates the number of points N on E_p.
- If Alice now wants to sign a message m, which we consider to be an integer modulo N, she first chooses a random positive integer k relatively prime to N.
- Alice then computes $Q_r = kQ_a = (x, y)$ and $s = k^{-1}(m - dx) \pmod N$, and sends Bob her signed message (m, Q_r, s).
- Bob verifies that Alice's signature is correct by computing $xQ_b + sQ_r$ and comparing it to mQ_a. If the results are equal, he accepts the signature, and otherwise he rejects it.
- The verification works because $xQ_b + sQ_r = x(dQ_a) + s(kQ_a) = (m - dx)Q_a = xdQ_a + mQ_a - dxQ_a = mQ_a$, where we are using the fact that $sk \equiv m - dx \pmod N$ to deduce that $ksQ_a = (m - dx)Q_a$ since the order of Q_a necessarily divides N.

- As with the elliptic-curve ElGamal encryption scheme, the security of this procedure ultimately relies on the difficulty of computing a discrete logarithm and the fact that k is randomly chosen.
  - It does not depend on the difficulty of computing the number of points on the curve N, which could even be published as part of the public key if desired.

- Example: Alice publishes her elliptic-curve ElGamal signature key with $E : y^2 = x^3 + 7x + 1$, $p = 44927$, $Q_a = (3174, 1067)$, and $Q_b = dQ_a = (38921, 25436)$ with her secret $d = 25661$. Bob then sends her the message $m = 17781$. Generate a signature for this message with $k = 33050$ and verify that it is correct.
  - Alice computes the number of points on the curve, $N = 44651$, which happens to be prime.
  - She then computes $Q_r = kQ_a = (11123, 34794) = (x, y)$ and $s = k^{-1}(m - dx) \equiv 42665 \pmod N$.
  - She then sends the pair (Q_r, s) to Bob, who then evaluates $xQ_b + sQ_r = (29063, 26534) + (36219, 42811) = (35670, 7590)$ and compares it to $mQ_a = (35670, 7590)$.
  - The results are equal, so Bob accepts the signature.

7.3 Rational and Integral Points on Elliptic Curves

- We now discuss the classical problems of finding rational and integral points on a given elliptic curve E. Such questions arise quite naturally in the context of solving Diophantine equations, and we will give some applications of these results to Diophantine equations.

7.3.1 Torsion Points on Elliptic Curves

- We first discuss the problem of finding rational points of small order on a given elliptic curve E in Weierstrass form: $y^2 = x^3 + Ax + B$: in other words, we are seeking the m-torsion points P with $mP = \infty$.
  - Before making any calculations, we observe that the m-torsion points form a subgroup of all points on E, since $m\infty = \infty$ and if $mP = \infty = mQ$ then $m(P - Q) = \infty$ as well.
  - This m-torsion subgroup of E is often denoted $E[m]$. When we want to emphasize the field K over which we are considering E, we will write this subgroup as $E_K[m]$.

- Now we can make some preliminary remarks about the structure of $E[m]$ for some small m.
  - Trivially, ∞ is the only point of order 1 on E.
  - The first nontrivial case is to identify the points of order 2: these points satisfy $P + P = \infty$. Geometrically, this means that if we consider the tangent line to the graph of E at P, then the third intersection point of P with E is the point at infinity.
  - It is not hard to see that this is equivalent to saying that the tangent line at P is vertical. From the explicit formula $2yy' = 3x^2 + A$ we see that this is, in turn, equivalent to saying that $y = 0$.

- Therefore, the points (x, y) of order 2 are those having $y = 0$. Since this requires $x^3 + Ax + B = 0$, we see that there are at most 3 such points.
- If we are searching for points over $\mathbb{C}$ (or another algebraically closed field), then there will be exactly 3 such points, since by assumption the elliptic curve is nonsingular so $x^3 + Ax + B$ has no repeated roots.
- Over arbitrary fields K, we may have a smaller number of roots of the cubic $x^3 + Ax + B$: it is possible that this cubic could have no K-rational points, 1 K-rational point, or 3 K-rational points (note that 2 points is not possible because if the cubic has two linear factors then it is a product of 3 linear factors).
- For points of order 3, we see that such points P satisfy $P + P + P = \infty$ so that $P + P = -P$, which means that the third intersection point of the tangent line to E at P also goes through P. Equivalently, this says that the point P is an inflection point of the curve.

- Example: Find the points of order 2 on the elliptic curve $E : y^2 = x^3 + x$ over $\mathbb{Q}$ and over $\mathbb{C}$, and identify the group structure of the 2-torsion group $E[2]$ over each field.
  - From the discussion above, the 2-torsion points are the points with $y = 0$, which requires $x^3 + x = 0$ so that $x = 0$ or $x = \pm i$.
  - Over $\mathbb{Q}$, there is therefore one 2-torsion point $\boxed{(0, 0)}$. Then the 2-torsion group $E_{\mathbb{Q}}[2]$ is $\{\infty, (0, 0)\}$ and its group structure is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
  - Over $\mathbb{C}$ we have three 2-torsion points: $\boxed{(0, 0),\ (i, 0),\ (-i, 0)}$. Then the 2-torsion group $E_{\mathbb{Q}}[2]$ is $\{\infty, (0, 0), (i, 0), (-i, 0)\}$. Since all of the nontrivial elements in this group have order 2, the group structure is isomorphic to the Klein 4-group $V_4 \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
  - Explicitly, here is the addition table for the 2-torsion points on E over $\mathbb{C}$:

$+$	∞	$(0,0)$	$(i,0)$	$(-i,0)$
∞	∞	$(0,0)$	$(i,0)$	$(-i,0)$
$(0,0)$	$(0,0)$	∞	$(-i,0)$	$(i,0)$
$(i,0)$	$(i,0)$	$(-i,0)$	∞	$(0,0)$
$(-i,0)$	$(-i,0)$	$(i,0)$	$(0,0)$	∞

- For points of higher order, it is even more difficult to give nice geometric or algebraic descriptions of $E[m]$.
  - One may try to compute explicitly the coordinate relations for these points; however, the resulting multiplication-by-m formulas end up being extremely complicated and unpleasant.
  - It is a rather long and convoluted (though not conceptually difficult) calculation to show that if $P = (x, y)$, then $mP = (x_m, y_m)$ where x is a rational function of degree $m^2 - 1$ in x (one may in fact eliminate y from all of these relations for the x-coordinates) and y_m is a rational function of degree m^2 in x and y.
  - Then $mP = \infty$ precisely when the denominator in the x-coordinate is equal to zero. Since this denominator polynomial in x has degree $m^2 - 1$, this means there are at most m^2 m-torsion points (note that ∞ must be added to the total).
  - One can also show that the denominator polynomial is separable over any field whose characteristic does not divide m, so it has distinct roots.
  - In particular, over the complex numbers $\mathbb{C}$, the m-torsion points form a group of order m^2, and thus over subfields of $\mathbb{C}$ (e.g., $\mathbb{Q}$) the m-torsion points will be a subgroup of the m-torsion group over $\mathbb{C}$.
  - Indeed, the fact that the set of complex m-torsion points has order m^2 for all m implies (by a straightforward induction on m using the structure theorem for finite abelian groups) that the group structure is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$ for all m. In general, over other algebraically closed fields, the group of m-torsion points will be isomorphic to a subgroup of $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$.
  - In particular, we obtain the nontrivial fact that the group of m-torsion points of E, over any field, is always generated by at most 2 generators.
  - Remark (for those who like complex analysis and topology): We will remark also that this structural fact over $\mathbb{C}$ can also be obtained by observing that the Riemann surface associated to a nonsingular elliptic curve $y^2 = x^3 + Ax + B$ is topologically a torus, and is diffeomorphic to $\mathbb{C}$ modulo a discrete lattice $\Lambda \cong \mathbb{Z}^2$. The torsion points then correspond to the elements of $\frac{1}{m}\Lambda/\Lambda$ in this quotient, which is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$.

7.3.2 Elliptic Curves Over $\mathbb{Q}$

- We now discuss the problem of computing rational points on elliptic curves. The following quite deep theorem establishes that the group of $\mathbb{Q}$-rational points on any elliptic curve E is always finitely generated:
- <u>Theorem</u> (Mordell): Let E be an elliptic curve over $\mathbb{Q}$. Then the group $E(\mathbb{Q})$ of rational points on E is finitely generated.
    - By the structure theorem for finitely generated abelian groups, this says $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E_{\mathrm{Tor}}(\mathbb{Q})$ where $E_{\mathrm{Tor}}(\mathbb{Q})$ is the set of $\mathbb{Q}$-torsion points of E (i.e., the set of $\mathbb{Q}$-rational points of E having finite order), which is a finite abelian group and thus is a direct sum of cyclic groups.
    - For any given elliptic curve E, the torsion subgroup $E_{\mathrm{Tor}}(\mathbb{Q})$ can be computed quite explicitly, as we will describe below.
    - The quantity r is called the <u>rank</u> of the elliptic curve, and is equal to the number of linearly-independent points one may construct on E. The rank is much more difficult to compute, and there is no known direct algorithm that is guaranteed to compute it (though in practice the rank of most curves can be computed).
    - It is not currently known whether elliptic curves over $\mathbb{Q}$ can have an arbitrarily large rank, and the historical consensus has switched back and forth between thinking ranks can be arbitrarily large and thinking that ranks are uniformly bounded above. Elkies has given a construction for an elliptic curve that has rank at least 28 (and it is expected this curve has rank exactly 28)[3]. It has been shown by Bhargava and Shankar in 2015 that the average rank (suitably defined) of an elliptic curve is at most 7/6: the actual average is expected to be 1/2 (with 50% of elliptic curves having rank 0 and 50% having rank 1, asymptotically).
    - The result of the Mordell-Weil theorem is relatively deep, and we will not go through all the calculations in the proof, but rather just outline the main ideas.
    - First, one proves the so-called "weak Mordell-Weil theorem": that for any positive integer m, the group $E(\mathbb{Q})/mE(\mathbb{Q})$ is finitely generated.
    - Of course, the weak Mordell-Weil theorem does not imply the full Mordell-Weil theorem directly, because there are many non-finitely-generated groups G such that G/mG is finitely generated (for example, $\mathbb{Q}$ and $\mathbb{R}$ both have $G/mG = 0$ for all m).
    - The difficulty is that knowing G/mG is finitely generated does not imply G is finitely generated, because G could contain many elements that are divisible by m. The task then is to eliminate this possibility, which can be done using the theory of heights: one defines a "height function", measuring roughly the complexity of a point on the curve, and then shows that the height of large multiples of a point tends to be larger than the height of the original point.
    - One such height function on points $(x, y) = (p_x/q_x, p_y/q_y)$ is $\max(\log p_x, \log q_x)$: essentially, the maximum number of digits appearing in the numerator or denominator of the coordinates.
    - Using heights, we can show that there are a bounded number of points in $E(\mathbb{Q})$ of height less than any fixed bound: thus, any point that is a multiple of m has to be "large" for large m.
    - By fine-tuning the details of this argument, we can deduce that a finite number of generators will suffice to generate the group $E(\mathbb{Q})$: the idea is to show that for any point P on E, we may subtract appropriate multiples of the coset representatives of the finite group $E(\mathbb{Q})/mE(\mathbb{Q})$ to obtain a new point whose height is bounded independently of P. Since there are then only finitely many such points, adding them to our list will yield a finite generating set for $E(\mathbb{Q})$.
- The following theorem of Nagell and Lutz provides a very convenient way to calculate the torsion points on any elliptic curve over $\mathbb{Q}$:
- <u>Theorem</u> (Nagell-Lutz): Suppose E is an elliptic curve over $\mathbb{Q}$ with Weierstrass equation $y^2 = x^3 + Ax + B$ where A and B are integers, and let $D = -4A^3 - 27B^2$ be the discriminant of E. If $P = (x, y)$ is a rational point of finite order, then x and y are integers. Furthermore, either $y = 0$ or y^2 divides D.

[3] The equation of Elkies' curve is $x^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x + 3448161179503055646703298569039072037485594435931918036126600829629193944873224342 9$

- We emphasize here that the Nagell-Lutz theorem is not an if-and-only-if: there can exist points (x, y) with y dividing D that do not have finite order.
- We will again only outline the ideas in the proof of the Nagell-Lutz theorem, rather than giving the full details.
- First, the idea is to show that if P has finite order, then its coordinates must be integers: we can do this by showing that it is not possible for any prime to divide the denominator of either coordinate.
- To establish this, we can use the same general idea as in the proof of Mordell's theorem: namely, consider what happens to the height of a point P under scaling.
- Instead of using the height function in Mordell's theorem, however, we use the so-called p-adic height, defined as follows: for any rational a/b we can pull out the factors of p to write $\frac{a}{b} = p^v \cdot \frac{m}{n}$ for some m, n not divisible by p; then we define the p-adic valuation as $\text{ord}_p(a/b) = v$.
- By analyzing the behavior of this valuation with respect to the group law on E, we can eventually show that it is not possible to have a point of finite order with negative p-adic valuation for any p, since the valuation of multiples of P would have to become arbitrarily large and negative.
- For the second part of the theorem (that $y = 0$ or y^2 divides D), suppose P has finite order. If $2P = \infty$ then as we observed earlier, $y = 0$. Otherwise assume $2P \neq 0$: then since $2P$ also has finite order, its coordinates are also integral.
- If $P = (a, b)$ and $2P = (c, d)$, then $c = m^2 - a$ and $d = -m(m^2 - 3a) - b$, with $m = \frac{3a^2 + A}{2b}$. Since $m^2 = a + c$ is an integer and m is rational, then m is an integer. This means $2b$ hence b divides $3a^2 + A$. But since $b^2 = a^3 + Aa + B$, we see that b^2 divides both $(3a^2 + A)^2$ and $a^3 + Aa + B$. By eliminating a from these relations, we can eventually conclude that b^2 divides D.

- The result of the Nagell-Lutz theorem gives us a very effective way to compute all of the torsion points on E.
  - First, we compute all of the possible torsion points: these are the integral points (x, y) on E where $y = 0$ or y^2 divides D, per the theorem above.
  - We then test whether these points have finite order. A priori, a rational point P could potentially have very large order, but since the torsion points form a subgroup and we have just listed all of the possible elements of this group, we have an upper bound on the possible order of the group and hence on the possible order of P.
  - More efficiently, to test whether P has finite order, we could simply compute the list $\{P, 2P, 3P, 4P, \dots\}$, or even just $\{P, 2P, 4P, 8P, \dots\}$: if any of the multiples of P fail to land on our list, then P cannot have finite order; otherwise, the multiples of P must necessarily repeat since our list is finite, in which case P (and all of its multiples) does have finite order.

- <u>Example</u>: Find the rational torsion points on the elliptic curve $E : y^2 = x^3 - 4x + 3$ and identify their group structure.
  - Here, we have $A = -4$ and $B = 3$, so the discriminant is $D = -4A^3 - 27B^2 = 13$.
  - Since D is squarefree, the only possible y-coordinates are 0 and ± 1.
  - Testing $y = 0$ (so that $x^3 - 4x + 3 = 0$) yields a single rational solution $x = 1$, giving a 2-torsion point $(1, 0)$.
  - Testing $y = \pm 1$ (so that $x^3 - 4x + 3 = \pm 1$) yields no rational solutions in either case, as the resulting cubic is irreducible.
  - Therefore, we see that there are two rational torsion points on E: $\boxed{(1, 0) \text{ and } \infty}$. The torsion group has order 2 and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

- <u>Example</u>: Find the rational torsion points on the elliptic curve $E : y^2 = x^3 + 16$ and identify their group structure.
  - Here, we have $A = 0$ and $B = 16$, so the discriminant is $D = -4A^3 - 27B^2 = -2^8 3^3$.
  - Then the possible y-coordinates are 0, ± 1, ± 2, ± 4, ± 8, ± 16, ± 3, ± 6, ± 12, ± 24, and ± 48.

- Testing each of these in turn yields two potential torsion points, namely, $(0, \pm 4)$.
- If we take $P = (0, 4)$ then we can compute $2P = (0, -4)$ and $3P = \infty$, so these points are indeed torsion points.
- Thus, there are three rational torsion points on E: $\boxed{(0, \pm 4) \text{ and } \infty}$. The torsion group has order 3 and is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

- Example: Find the rational torsion points on the elliptic curve $E : y^2 = x^3 - 2x + 1$ and identify their group structure.
  - Here, we have $A = -2$ and $B = 1$, so the discriminant is $D = -4A^3 - 27B^2 = 5$.
  - Then the possible y-coordinates are 0 and ± 1. Testing yields the potential torsion points $(1, 0)$, $(0, \pm 1)$.
  - If we take $P = (0, 1)$ then we can compute $2P = (1, 0)$, $3P = (0, -1)$, and then $4P = \infty$, so all of these points are indeed torsion points.
  - Thus, there are four rational torsion points on E: $\boxed{(0, \pm 1),\ (1, 0), \text{ and } \infty}$. The torsion group has order 4 and is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

- Example: Find the rational torsion points on the elliptic curve $E : y^2 = x^3 - 351x + 1890$ and identify their group structure.
  - Here, we have $A = -351$ and $B = 1890$, so the discriminant is $D = -4A^3 - 27B^2 = 2^4 3^{14}$.
  - Then the possible y-coordinates are 0 and $\pm 2^a 3^b$ for $a \in \{0, 1, 2\}$ and $b \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.
  - If $y = 0$ then we obtain three 2-torsion points, namely $(-21, 0)$, $(6, 0)$, $(15, 0)$.
  - For the other 24 possible values of y, some computation yields four additional candidate points: $(-3, \pm 54)$ and $(33, \pm 162)$.
  - With $P = (33, 162)$ we can compute $2P = (15, 0)$, $3P = (33, -162)$, and $4P = \infty$, so this point has order 4.
  - Likewise, with $Q = (-3, 54)$ we can compute $2Q = (15, 0)$, $3Q = (-3, -54)$, and $4Q = \infty$, so this point also has order 4.
  - Thus, there are eight rational torsion points on E: $\boxed{(-3, \pm 54),\ (33, \pm 162),\ (-21, 0),\ (6, 0),\ (15, 0), \text{ and } \infty}$. The torsion group has order 8 and is isomorphic to $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, where we can take (a, b) mapping to $aP + b(Q - P)$.

- We can also use the Nagell-Lutz theorem to establish that a given point has infinite order on E.
  - Most obviously, if the point does not have integral coordinates, then it is not a torsion point. Even if its coordinates are integral, if its y-coordinate is nonzero and its square does not divide D, then the point cannot be a torsion point.
  - Furthermore, even if all of these conditions are satisfied, if we compute $2P, 3P, 4P, \dots$ and any of these points have non-integral coordinates or have a nonzero y-coordinate with y^2 not dividing D, then P must have infinite order.

- Example: Show that the elliptic curve $E : y^2 = x^3 + 2$ has infinitely many rational points.
  - Testing small values of x reveals two integral points: $(x, y) = (-1, \pm 1)$.
  - If we take $P = (-1, -1)$, then P could be a torsion point, since its y-coordinate -1 has its square dividing the discriminant $D = -108$.
  - However, we can calculate $2P = (17/4, 71/8)$, and so since $2P$ does not have integral coordinates, it is not a torsion point, and thus neither is P.
  - This means that P has infinite order, which is to say, all of the points $P, 2P, 3P, 4P, \dots$ are distinct. Since these all have rational coordinates, we see that E has infinitely many rational points.
  - Remark: It is much harder to prove, but in fact E has rank 1 and its group of rational points is generated by P.

- It follows from the Nagell-Lutz theorem that the group of rational torsion points on an elliptic curve is always finite.
  - Although it may seem that the group could potentially be arbitrarily large, in fact, it cannot have order greater than 16.
  - The following quite deep theorem of Mazur establishes that there is a fairly small list of possible torsion groups:
- Theorem (Mazur): If E is an elliptic curve, then the number of rational torsion points (including ∞) can be any integer from 1 to 12 inclusive, excluding 11, or 16. More explicitly, there are 15 possible group structures for the rational torsion points: the trivial group (order 1), $\mathbb{Z}/2\mathbb{Z}$ (order 2), $\mathbb{Z}/3\mathbb{Z}$ (order 3), $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}/4\mathbb{Z}$ (order 2), $\mathbb{Z}/5\mathbb{Z}$ (order 5), $\mathbb{Z}/6\mathbb{Z}$ (order 6), $\mathbb{Z}/7\mathbb{Z}$ (order 7), $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ or $\mathbb{Z}/8\mathbb{Z}$ (order 8), $\mathbb{Z}/9\mathbb{Z}$ (order 9), $\mathbb{Z}/10\mathbb{Z}$ (order 10), $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ or $\mathbb{Z}/12\mathbb{Z}$ (order 12), or $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})$ (order 16).
  - The proof of this theorem involves quite advanced methods: the idea is to study the points on various modular curves and use a (tremendous!) amount of case analysis to eliminate all of the other possible torsion orders and other possible group structures.
  - There also exist infinite families of elliptic curves having each of the groups listed as its torsion group.
- In some situations (e.g., if we are solving a Diophantine equation) we are often interested particularly in the integral points on an elliptic curve.
  - As we have remarked, an elliptic curve of positive rank necessarily has infinitely many rational points.
  - However, the following result of Siegel establishes that only finitely many of these rational points can be integral:
- Theorem (Siegel): If E is a (nonsingular) elliptic curve over $\mathbb{Q}$, then E has only finitely many integral points.
  - We emphasize here that E must be nonsingular, since (for example) the singular curve $y^2 = x^3$ has infinitely many integral points, namely $(x, y) = (n^2, n^3)$ for any integer n.
  - Siegel's original proof, like the proof of Mordell's theorem, is ineffective, in the sense that it does not give an explicit bound on the possible size of the integral points in terms of the coefficients of E.
  - For certain curves, the results can be made explicit using results of Baker on linear forms in logarithms, but the results typically are still computationally infeasible in practice.
  - For example, one such result says that if (x, y) is an integral point on $y^2 = x^3 + ax^2 + bx + c$, then $\max(|x|, |y|) \leq \exp\left[(1{,}000{,}000 \max(|a|, |b|, |c|))^{1{,}000{,}000}\right]$. Even for quite small a, b, c, this bound is completely infeasible to work with.
- For certain special curves, such as $x^3 - by^3 = c$, one can establish better results, using Diophantine approximation ideas similar to those we used in studying Pell's equation $x^2 - Dy^2 = r$.
  - Explicitly, as shown by Thue, if b is a positive integer that is not a cube and C is any fixed positive constant, then there are only finitely many rational numbers p/q such that $\left|p/q - \sqrt[3]{b}\right| < C/q^3$.
  - In fact, Thue showed that there are only finitely many p/q satisfying $\left|p/q - \sqrt[3]{b}\right| < C/q^{5/2+\epsilon}$ for any $\epsilon > 0$, and this result has been improved by Roth to show that there are only finitely many p/q satisfying $\left|p/q - \sqrt[3]{b}\right| < C/q^{2+\epsilon}$ for any $\epsilon > 0$. Since (as we showed via continued fractions) there are infinitely many p/q with $|p/q - \alpha| < C/q^2$ for any irrational α, Roth's result is essentially the best possible.
  - Combining Thue's result with the straightforward estimate that if $x^3 - by^3 = c$ then $\left|x/y - \sqrt[3]{b}\right| \leq \dfrac{4|c|}{3b^{2/3}} \cdot \dfrac{1}{|y|^3}$, which is of the form above with $C = 4|c|/(3b^{2/3})$, implies immediately that there are only finitely many integral pairs (x, y) with $x^3 - by^3 = c$.
- Various computational improvements have been made that allow efficient calculation of integral points on most elliptic curves.

- Such algorithms are implemented in some algebra packages such as Sage and Magma, and a large number of elliptic curves have been tabulated in various databases such as the L-Functions and Modular Forms Database (LMFDB). Using these, one may generate examples of elliptic curves having relatively small coefficients that have quite a few integral points.
- For example, the curve $E : y^2 = x^3 - 1267x + 17230$ has 82 integral points, as follows: $(-41, \pm 16)$, $(-37, \pm 116)$, $(-33, \pm 152)$, $(-29, \pm 172)$, $(-17, \pm 184)$, $(-10, \pm 170)$, $(-1, \pm 136)$, $(3, \pm 116)$, $(11, \pm 68)$, $(15, \pm 40)$, $(18, \pm 16)$, $(19, \pm 4)$, $(22, \pm 2)$, $(23, \pm 16)$, $(27, \pm 52)$, $(31, \pm 88)$, $(34, \pm 116)$, $(47, \pm 248)$, $(51, \pm 292)$, $(54, \pm 326)$, $(87, \pm 752)$, $(107, \pm 1052)$, $(115, \pm 1180)$, $(151, \pm 1808)$, $(239, \pm 3656)$, $(279, \pm 4624)$, $(363, \pm 6884)$, $(418, \pm 8516)$, $(491, \pm 10852)$, $(515, \pm 11660)$, $(703, \pm 18616)$, $(1167, \pm -39848)$, $(1362, \pm 50248)$, $(3967, \pm 249848)$, $(4559, \pm 307816)$, $(6623, \pm 538984)$, $(14006, \pm 1657562)$, $(18127, \pm 2440552)$, $(42331, \pm 8709388)$, $(77169, \pm 21624796)$, $(878838, \pm 823878634)$. It can be shown that the group of rational points on this curve is isomorphic to $\mathbb{Z}^4$, and is generated by the four points $(15, 40)$, $(19, 4)$, $(23, 16)$, and $(31, 88)$. (The difficulty is in proving that these four points are linearly independent.)
- In contrast, the curve $E : y^2 = x^3 - 1267x + 17231$, which differs only by 1 in the constant term, has no integral points at all, while the curve $E : y^2 = x^3 - 1266x + 17230$, differing by 1 in the linear term, has two integral points $(5, \pm 105)$.
- As another example, the curve $E : y^2 = x^3 - 1386747x + 368636886$ is the curve with the smallest discriminant in this Weierstrass form whose $\mathbb{Q}$-torsion group is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})$. It can also be shown to have rank 0, so in fact its full group of $\mathbb{Q}$-rational points is $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})$. The integral points on this curve are therefore just its torsion points, which (along with ∞) are $(-1293, 0)$, $(-933, \pm 29160)$, $(-285, \pm 27216)$, $(147, \pm 12960)$, $(282, 0)$, $(1011, 0)$, $(1227, \pm 22680)$, $(2307, \pm 97200)$, and $(8787, \pm 816480)$. Explicitly, one can verify that $P = (-933, 29160)$ has order 8, and that $4P = (1011, 0)$. Thus, P and $Q = (282, 0)$ generate the group of rational points on E.

7.3.3 The Congruent Number Problem

- We will finish our discussion of elliptic curves with a brief examination of a famous classical number theory problem that turns out to reduce to the question of whether an elliptic curve has a nontrivial rational point.
- Definition: We say a positive integer n is a congruent number if there exists a right triangle with rational side lengths whose area is n.
  - Example: 6 is a congruent number, since it is the area of a 3-4-5 right triangle.
  - Example: 5 is a congruent number: although 5 cannot be the area of a right triangle with integer side lengths, it is the area of a triangle with side lengths 3/2, 20/3, and 41/6 (which is similar to the integer-sided 9-40-41 triangle).
  - Example: No square is a congruent number (thus in particular, 1 and 4 are not congruent numbers). If it were true that $\frac{1}{2}ab = k^2$ and $a^2 + b^2 = c^2$, then $c^2 + 4k^2 = (a + b)^2$ and $c^2 - 4k^2 = (a - b)^2$, so multiplying these equations would yield $c^4 - (2k)^4 = (a^2 - b^2)^2$, which is equivalent to the equation $c^4 = d^4 + e^2$. By clearing denominators and then using essentially the same infinite descent argument as for the Diophantine equation $c^4 + d^4 = e^2$, we can show that there is no solution to this congruence in positive integers, and thus no square can be a congruent number.
  - From similarity, it is easy to see that n is a congruent number if and only if $k^2 n$ is a congruent number for any positive integer k. Also, from our characterization of Pythagorean triples, we can see that the area of any right triangle with integer side lengths is of the form $k^2 st(s^2 - t^2)$: thus, if we take out the square factors, we are equivalently searching for integers that are the squarefree part of $st(s^2 - t^2)$ for some s and t.
  - Although it might seem that congruent numbers would be easy to enumerate from this procedure, the squarefree part of $st(s^2 - t^2)$ varies greatly even for s, t of similar sizes. For example, $(s, t) = (5, 4)$ gives $st(s^2 - t^2) = 5 \cdot 4 \cdot 9$ with a squarefree part of 5, while $(s, t) = (5, 2)$ gives $st(s^2 - t^2) = 5 \cdot 2 \cdot 3 \cdot 7 = 210$ with a squarefree part of 210.
  - There are many other ways to characterize congruent numbers: if the legs of the right triangle are a, b and the hypotenuse is c, then we want solutions to the system $ab = 2n$ and $a^2 + b^2 = c^2$. These equations imply $c^2 + 4n = (a + b)^2$ and $c^2 - 4n = (a - b)^2$, so if we set $s = a + b$ and $d = a - b$ we equivalently have $(c/2)^2 + n = (s/2)^2$ and $(c/2)^2 - n = (d/2)^2$.

- Since $(c/2)^2$ is also a square, the above calculations show that n is a congruent number if and only if there exists an arithmetic progression $x-n=(a-b)^2/4$, $x=c^2/4$, $x+n=(a+b)^2/4$ of nonzero rational squares having common difference n.
- If we multiply these conditions, this means (equivalently) that the product $x(x-n)(x+n)=x^3-n^2x$ must also be the square of some nonzero rational number y: thus, if n is a congruent number, we must have a rational point on the elliptic curve $E_n : y^2=x^3-n^2x$ with $y \neq 0$ (equivalently, not a 2-torsion point).

- In fact, the converse of this statement is true as well:
- <u>Proposition</u> (Congruent Numbers): The positive integer n is a congruent number if and only if the elliptic curve $E : y^2=x^3-n^2x$ has a rational point with $y \neq 0$.
  - <u>Motivation</u>: If we follow through the algebra above, we can see that we can take $x=n\dfrac{(a+c)}{b}$ and $y=2n^2\dfrac{(a+c)}{b^2}$. If we invert these calculations we can rederive the values of a,b,c from n,x,y as $a=\dfrac{y}{x}$, $b=\dfrac{2nx}{y}$, and $c=\dfrac{2x^2}{y}-\dfrac{y}{x}=\dfrac{x^2+n^2}{y}$, which we then just have to show will work.
  - <u>Proof</u>: Clearly, if (x,y) is a rational point on E with $y \neq 0$, then $x \neq 0, \pm n$.
  - First, if (a,b,c) has $a^2+b^2=c^2$ and $ab=2n$, then for $x=n\dfrac{(a+c)}{b}$ and $y=2n^2\dfrac{(a+c)}{b^2}$ we can see $x=\dfrac{1}{2}a(a+c)$ and $y=\dfrac{1}{2}a^2(a+c)$.
  - Then $y^2/x=\dfrac{1}{2}a^3(a+c)$, while $x^2-n^2=\dfrac{1}{4}a^2(a+c)^2-\dfrac{1}{4}a^2b^2=\dfrac{1}{4}a^2(2a^2+2ac)=\dfrac{1}{2}a^3(a+c)$. Thus $y^2/x=x^2-n^2$, so $y^2=x^3-n^2x$, as claimed. We therefore obtain a rational point on E with $y \neq 0$ as claimed.
  - Conversely, suppose that $y^2=x^3-n^2x$ has $y \neq 0$ so that $x \neq 0$ also, and then set $a=\dfrac{y}{x}$, $b=\dfrac{2nx}{y}$, and $c=\dfrac{2x^2}{y}-\dfrac{y}{x}=\dfrac{2x^3-y^2}{xy}=\dfrac{x^2+n^2}{y}$. Note that a,b,c are well-defined, nonzero rational numbers since $x,y \neq 0$.
  - Then clearly we have $\dfrac{1}{2}ab=n$, and we also have $(c-b)(c+b)=\dfrac{(x-n)^2}{y}\cdot\dfrac{(x+n)^2}{y}=\dfrac{(x^2-n^2)^2}{x(x^2-n^2)}=\dfrac{x(x^2-n^2)}{x^2}=\dfrac{y^2}{x^2}=a^2$, meaning that $a^2+b^2=c^2$ as required.
  - We can then replace any of a,b,c with their absolute values without affecting these conditions, and then we see n is a congruent number, as claimed.

- It can be shown that the only torsion points on the <u>congruent-number elliptic curve</u> $E_n : y^2=x^3-n^2x$ are the 2-torsion points ∞, $(0,0)$, and $(\pm n, 0)$.
  - One way to do this is to observe that the reduction-mod-p map from the torsion points of E_n (which have integer coordinates) to the $\mathbb{F}_p$-points of E_n modulo p is a group homomorphism, and that it is injective whenever p does not divide the discriminant of E_n. The first part follows essentially from the observation that the definition of the group law is the same over $\mathbb{Q}$ and over $\mathbb{F}_p$, while the second part follows from noting that no nontrivial torsion point can reduce to ∞ modulo p when p does not divide the discriminant of E, since its denominator cannot be zero.
  - Next, one observes that E_n always has exactly $p+1$ points over $\mathbb{F}_p$ when p is a prime congruent to 3 modulo 4. Finally, since the reduction-mod-p map must be injective for sufficiently large p, one then uses the fact that there are arbitrarily large primes lying in any residue class a modulo m with a relatively prime to m to select various primes p for which the greatest common divisor of the values p_i+1 is 4. Putting all of this together establishes that the size of the torsion subgroup of E over $\mathbb{Q}$ must have order dividing 4, and since there are in fact four 2-torsion points, there cannot be any other torsion points.

- Thus, since the only torsion points on E_n have $y = 0$, we see that there is a rational point on E_n with $y \neq 0$ if and only if E_n has rank at least 1: then by negating if necessary, we obtain a rational point with $y > 0$. Since we just showed that this condition is equivalent to saying that n is a congruent number, we deduce that n is a congruent number precisely when E_n has rank at least 1.

- One can then compute the rank of E_n for a given n to establish whether or not n is a congruent number.
  - For example, for $n = 1, 2, 3, 4$, the rank is 0, so these are not congruent numbers. But for $n = 5$ we have a rational point $(x, y) = (-4, 6)$, which yields $(a, b, c) = (-3/2, -20/3, 41/6)$, which (up to sign) is the triangle of area 5 we identified earlier.
  - For $n = 7$ we can find a rational point $(x, y) = (25, 120)$, which yields $(a, b, c) = (24/5, 35/12, 337/60)$, which indeed yields a right triangle having area 7.
  - Much work has been done in classifying congruent numbers, but as of 2021, a full characterization is still not known. It has been shown that if p is a prime congruent to 3 modulo 8, then p is not a congruent number, while if p is a prime congruent to 5 or 7 modulo 8, then p is a congruent number.

- A 1983 theorem of Tunnell, which relies quite heavily on modular forms, gives an efficient way to determine whether n is a congruent number.

- Theorem (Tunnell): If n is an odd congruent number then the number of solutions in integers to $n = 2x^2 + y^2 + 32z^2$ is equal to half the number of solutions of $n = 2x^2 + y^2 + 8z^2$, while if n is even then the number of solutions to $n/2 = 4x^2 + y^2 + 32z^2$ is equal to half the number of solutions of $n/2 = 4x^2 + y^2 + 8z^2$.
  - Tunnell also showed that if the weak Birch/Swinnerton-Dyer conjecture (which states that the algebraic rank r of an elliptic curve is equal to the "analytic rank", which is the order of vanishing of the L-function associated to the elliptic curve at $s = 1$) holds for E_n, then the converse of the criterion above also holds (and thus, it is an if-and-only-if condition).
  - Tunnell's theorem gives a fairly rapid way to show that particular n are not congruent numbers.
  - Example: If $n = 2$ then there are two solutions to $n/2 = 4x^2 + y^2 + 32z^2$ (namely, $(0, \pm 1, 0)$) and also two solutions to $n/2 = 4x^2 + y^2 + 8z^2$ (namely, $(0, \pm 1, 0)$). Thus, 2 is not a congruent number.
  - Example: If $n = 3$ then there are four solutions to $n = 2x^2 + y^2 + 32z^2$ (namely, $(\pm 1, \pm 1, 0)$ and also four solutions to $n = 2x^2 + y^2 + 8z^2$ (also $(\pm 1, \pm 1, 0)$). Thus, 3 is not a congruent number.
  - Example: If $n = 13$ then there are no solutions to $n = 2x^2 + y^2 + 32z^2$ or to $n = 2x^2 + y^2 + 8z^2$. This suggests n is in fact a congruent number, and indeed, searching for rational points on $y^2 = x^3 - 13^2 x$ will eventually identify the point $(x, y) = (-36/25, 1938/125)$, which yields the triangle sides $(a, b, c) = (323/30, 780/323, 106921/9690)$. Thus, 13 is a congruent number.

Well, you're at the end of my handout. Hope it was helpful.