Differential Equations (part 3): Systems of First-Order Differential Equations (by Evan Dummit, 2016, v. 2.00)

Contents

6 Systems of First-Order Linear Differential Equations

In many (perhaps most) applications of differential equations, we have not one but several quantities which change over time and interact with one another. Examples include the populations of the various species in an ecosystem, the concentrations of molecules involved in a chemical reaction, the motion of objects in a physical system, and the availability and production of items (goods, labor, materials) in economic processes.

In this chapter, we will outline the basic theory of systems of differential equations. As with the other differential equations we have studied, we cannot solve arbitrary systems in full generality: in fact it is very difficult even to solve individual nonlinear differential equations, let alone a system of nonlinear equations. We will therefore restrict our attention to systems of linear differential equations: as with our study of higher-order linear differential equations, there is an underlying vector-space structure to the solutions which we will explain. We will discuss how to solve many examples of homogeneous systems having constant coefficients.

6.1 General Theory of (First-Order) Linear Systems

- Before we start our discussion of systems of linear differential equations, we first observe that we can reduce any system of linear differential equations to a system of *first-order* linear differential equations (in more variables): if we define new variables equal to the higher-order derivatives of our old variables, then we can rewrite the old system as a system of first-order equations.
- Example: Convert the single 3rd-order equation $y''' + y' = 0$ to a system of first-order equations.
  - If we define new variables $z = y'$ and $w = y'' = z'$, then the original equation tells us that $y''' = -y'$, so $w' = y''' = -y' = -z$.
  - Thus, this single 3rd-order equation is equivalent to the first-order system $y' = z$, $z' = w$, $w' = -z$.
- Example: Convert the system $y_1'' + y_1 - y_2 = 0$ and $y_2'' + y_1' + y_2' \sin(x) = e^x$ to a systen of first-order equations.
  - If we define new variables $z_1 = y_1'$ and $z_2 = y_2'$, then $z_1' = y_1'' = -y_1 + y_2$ and $z_2' = y_2'' = e^x - y_1' - y_2' \sin(x) = e^x - z_1 - z_2 \sin(x)$.
  - So this system is equivalent to the first-order system $y_1' = z_1$, $y_2' = z_2$, $z_1' = -y_1 + y_2$, $z_2' = e^x - z_1 - z_2 \sin(x)$.
- Thus, whatever we can show about solutions of systems of first-order linear equations will carry over to arbitrary systems of linear differential equations. So we will talk only about systems of first-order linear differential equations from now on.
- Definition: The standard form of a system of first-order linear differential equations with unknown functions $y_1, y_2, \ldots, y_n$ is

$$\begin{aligned} y_1' &= a_{1,1}(x) \cdot y_1 + a_{1,2}(x) \cdot y_2 + \cdots + a_{1,n}(x) \cdot y_n + q_1(x) \\ y_2' &= a_{2,1}(x) \cdot y_1 + a_{2,2}(x) \cdot y_2 + \cdots + a_{2,n}(x) \cdot y_n + q_2(x) \\ \vdots & \quad \vdots \\ y_n' &= a_{n,1}(x) \cdot y_1 + a_{n,2}(x) \cdot y_2 + \cdots + a_{n,n}(x) \cdot y_n + q_n(x) \end{aligned}$$

  for some functions $a_{i,j}(x)$ and $q_i(x)$ for $1 \le i, j \le n$.

- We can write this system more compactly using matrices: if $A = \begin{bmatrix} a_{1,1}(x) & \cdots & a_{1,n}(x) \\ \vdots & \ddots & \vdots \\ a_{n,1}(x) & \cdots & a_{n,n}(x) \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} q_1(x) \\ \vdots \\ q_n(x) \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}$ so that $\mathbf{y}' = \begin{bmatrix} y_1'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}$, we can write the system more compactly as $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$.
- We say that the system is <u>homogeneous</u> if $\mathbf{q} = 0$, and it is <u>nonhomogeneous</u> otherwise.
- Most of the time we will be dealing with systems with constant coefficients, in which the entries of A are constant functions.
- An <u>initial condition</u> for this system consists of n pieces of information: $y_1(x_0) = b_1$, $y_2(x_0) = b_2$, ... , $y_n(x_0) = b_n$, where x_0 is the starting value for x and the b_i are constants. Equivalently, it is a condition of the form $\mathbf{y}(x_0) = \mathbf{b}$ for some vector $\mathbf{b}$.

* We also have a version of the Wronskian in this setting for checking whether function vectors are linearly independent:

* <u>Definition</u>: Given n vectors $\mathbf{v}_1 = \begin{bmatrix} y_{1,1}(x) \\ \vdots \\ y_{1,n}(x) \end{bmatrix}, \cdots, \mathbf{v}_n = \begin{bmatrix} y_{n,1}(x) \\ \vdots \\ y_{n,n}(x) \end{bmatrix}$ of length n with functions as entries, their <u>Wronskian</u> is defined as the determinant $W = \begin{vmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{vmatrix}$.
  - By our results on row operations and determinants, we immediately see that n function vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of length n are linearly independent if their Wronskian is not the zero function.

* Many of the theorems about general systems of first-order linear equations are very similar to the theorems about nth order linear equations.

* <u>Theorem</u> (Existence-Uniqueness): For a system of first-order linear differential equations, if the coefficient functions $a_{i,j}(x)$ and nonhomogeneous terms $p_j(x)$ are each continuous in an interval around x_0 for all $1 \le i, j \le n$, then the system

$$\begin{aligned} y_1' &= a_{1,1}(x) \cdot y_1 + a_{1,2}(x) \cdot y_2 + \cdots + a_{1,n}(x) \cdot y_n + p_1(x) \\ y_2' &= a_{2,1}(x) \cdot y_1 + a_{2,2}(x) \cdot y_2 + \cdots + a_{2,n}(x) \cdot y_n + p_2(x) \\ \vdots & \quad \vdots \\ y_n' &= a_{n,1}(x) \cdot y_1 + a_{n,2}(x) \cdot y_2 + \cdots + a_{n,n}(x) \cdot y_n + p_n(x) \end{aligned}$$

with initial conditions $y_1(x_0) = b_1$, ..., $y_n(x_0) = b_n$ has a unique solution $(y_1, y_2, \cdots, y_n)$ on that interval.
  - This theorem is not trivial to prove and we will omit the proof.
  - <u>Example</u>: The system $y' = e^x \cdot y + \sin(x) \cdot z$, $z' = 3x^2 \cdot y$ has a unique solution for every initial condition $y(x_0) = b_1$, $z(x_0) = b_2$.

* <u>Proposition</u>: Suppose $\mathbf{y}_{par}$ is one solution to the matrix system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Then the general solution $\mathbf{y}_{gen}$ to this equation may be written as $\mathbf{y}_{gen} = \mathbf{y}_{par} + \mathbf{y}_{hom}$, where $\mathbf{y}_{hom}$ is a solution to the homogeneous system $\mathbf{y}' = A\mathbf{y}$.
  - <u>Proof</u>: Suppose that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions to the general equation. Then $(\mathbf{y}_2 - \mathbf{y}_1)' = \mathbf{y}_2' - \mathbf{y}_1' = (A\mathbf{y}_1 + \mathbf{q}) - (A\mathbf{y}_2 + \mathbf{q}) = A(\mathbf{y}_1 - \mathbf{y}_2)$, meaning that their difference $\mathbf{y}_2 - \mathbf{y}_1$ is a solution to the homogeneous equation.

- Theorem (Homogeneous Systems): If the coefficient functions $a_{i,j}(x)$ are continuous on an interval I for each $1 \le i, j \le n$, then the set of solutions $\mathbf{y}$ to the homogeneous system $\mathbf{y}' = A\mathbf{y}$ on I is an n-dimensional vector space.
  - Proof: First, the solution space is a subspace, since it satisfies the subspace criterion:
    - [S1]: The zero function is a solution.
    - [S2]: If $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions, then $(\mathbf{y}_1+\mathbf{y}_2)' = \mathbf{y}_1' + \mathbf{y}_2' = A(\mathbf{y}_1+\mathbf{y}_2)$ so $\mathbf{y}_1+\mathbf{y}_2$ is also a solution.
    - [S3]: If α is a scalar and $\mathbf{y}$ is a solution, then $(\alpha\mathbf{y})' = \alpha\mathbf{y}' = \alpha(A\mathbf{y}) = A(\alpha\mathbf{y})$ so $\alpha\mathbf{y}$ is also a solution.
  - Now we need to show that the solution space is n-dimensional. We will do this by finding a basis.
    - Choose any x_0 in I. By the existence part of the existence-uniqueness theorem, for each $1 \le i \le n$ there exists a function $\mathbf{z}_i$ such that $\mathbf{z}_i(x_0)$ is the ith unit coordinate vector of $\mathbb{R}^n$, with $z_{i,i}(x_0) = 1$ and $x_{i,j}(x_0)$ for all $j \ne i$.
    - The functions $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ are linearly independent because their Wronskian matrix evaluated at $x = x_0$ is the identity matrix. (In particular, the Wronskian is not the zero function.)
    - Now suppose $\mathbf{y}$ is any solution to the homogeneous equation, with $\mathbf{y}(x_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.
    - Then the function $\mathbf{z} = c_1\mathbf{z}_1 + c_2\mathbf{z}_2 + \cdots + c_n\mathbf{z}_n$ also has $\mathbf{z}(x_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and is a solution to the homogeneous equation.
    - But by the uniqueness part of the existence-uniqueness theorem, there is only one such function, so we must have $\mathbf{y}(x) = \mathbf{z}(x)$ for all x: therefore $\mathbf{y} = c_1\mathbf{z}_1 + c_2\mathbf{z}_2 + \cdots + c_n\mathbf{z}_n$, meaning that $\mathbf{y}$ is in the span of $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$.
    - This is true for any solution function $\mathbf{y}$, so $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ span the solution space. Since they are also linearly independent, they form a basis of the solution space, and because there are n of them, we see that the solution space is n-dimensional.
- If we combine the above results, we can write down a fairly nice form for the solutions of a general system of first-order differential equations:
- Corollary: The general solution to the nonhomogeneous equation $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ has the form $\mathbf{y} = \mathbf{y}_{par} + C_1\mathbf{z}_1 + C_2\mathbf{z}_2 + \cdots + C_n\mathbf{z}_n$, where $\mathbf{y}_{par}$ is any one particular solution of the nonhomogeneous equation, $\mathbf{z}_1, \dots, \mathbf{z}_n$ are a basis for the solutions to the homogeneous equation, and $C_1, \dots, C_n$ are arbitrary constants.
  - This corollary says that, in order to find the general solution, we only need to find one function which satisfies the nonhomogeneous equation, and then solve the homogeneous equation.

6.2 Eigenvalue Method (Nondefective Coefficient Matrices)

- We now restrict our discussion to homogeneous first-order systems with constant coefficients: those of the form

$$\begin{aligned} y_1' &= a_{1,1}y_1 + a_{1,2}y_2 + \cdots + a_{1,n}y_n \\ y_2' &= a_{2,1}y_1 + a_{2,2}y_2 + \cdots + a_{2,n}y_n \\ \vdots & \quad \vdots \\ y_n' &= a_{n,1}y_1 + a_{n,2}y_2 + \cdots + a_{n,n}y_n \end{aligned}$$

which we will write in matrix form as $\mathbf{y}' = A\mathbf{y}$ with $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$.

* Our starting point for solving such systems is to observe that if $\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is an eigenvector of A with eigenvalue λ, then $\mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} e^{\lambda x}$ is a solution to $\mathbf{y}' = A\mathbf{y}$.
    * This follows simply by differentiating $\mathbf{y} = e^{\lambda x}\mathbf{v}$ with respect to x: we see $\mathbf{y}' = \lambda e^{\lambda x}\mathbf{v} = \lambda\mathbf{y} = A\mathbf{y}$.
    * In the event that A has n linearly independent eigenvectors, we will therefore obtain n solutions to the differential equation.
    * If these solutions are linearly independent, then since we know the solution space is n-dimensional, we would be able to conclude that our solutions are a basis for the solution space.
* Theorem (Eigenvalue Method): If A has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then the general solution to the matrix differential system $\mathbf{y}' = A\mathbf{y}$ is given by $\mathbf{y} = C_1 e^{\lambda_1 x}\mathbf{v}_1 + C_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_2$, where $C_1, \cdots, C_n$ are arbitrary constants.
    * Recall that if λ is a root of the characteristic equation k times, we say that λ has multiplicity k. If the eigenspace for λ has dimension less than k, we say that λ is "defective". The theorem allows us to solve the matrix differential system for any nondefective matrix.
    * Proof: By the observation above, each of $e^{\lambda_1 x}\mathbf{v}_1$, $e^{\lambda_2 x}\mathbf{v}_2$, $\cdots$, $e^{\lambda_n x}\mathbf{v}_n$ is a solution to $\mathbf{y}' = A\mathbf{y}$. We claim that they are a basis for the solution space.
    * To show this, we know by our earlier results that the solution space of the system $\mathbf{y}' = A\mathbf{y}$ is n-dimensional: thus, if we can show that these solutions are linearly independent, we would be able to conclude that our solutions are a basis for the solution space.
    * We can compute the Wronskian of these solutions: after factoring out the exponentials from each column, we obtain $W = e^{(\lambda_1 + \cdots + \lambda_n)x} \det(M)$, where $M = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & | & | \end{bmatrix}$.
    * The exponential is always nonzero and the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are (by hypothesis) linearly independent, meaning that $\det(M)$ is nonzero. Thus, W is nonzero, so $e^{\lambda_1 x}\mathbf{v}_1$, $e^{\lambda_2 x}\mathbf{v}_2$, $\cdots$, $e^{\lambda_n x}\mathbf{v}_n$ are linearly independent.
    * Since these solutions are therefore a basis for the solution space, we immediately conclude that the general solution to $\mathbf{y}' = A\mathbf{y}$ has the form $\mathbf{y} = C_1 e^{\lambda_1 x}\mathbf{v}_1 + C_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_2$, for arbitrary constants $C_1, \cdots, C_n$.
* The theorem allows us to solve all homogeneous systems of linear differential equations whose coefficient matrix A has n linearly independent eigenvectors. (Such matrices are called nondefective matrices.)
* Example: Find all functions y_1 and y_2 such that $\begin{matrix} y_1' & = & y_1 - 3y_2 \\ y_2' & = & y_1 + 5y_2 \end{matrix}$.
    * The coefficient matrix is $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$, whose characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-1 & 3 \\ -1 & t-5 \end{vmatrix} = (t-1)(t-5) + 3 = t^2 - 6t + 8 = (t-2)(t-4)$.
    * Thus, the eigenvalues of A are $\lambda = 2, 4$.
    * For $\lambda = 2$, we want to find the nullspace of $\begin{bmatrix} 2-1 & 3 \\ -1 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, so the 2-eigenspace is 1-dimensional and is spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

- For $\lambda = 4$, we want to find the nullspace of $\begin{bmatrix} 4-1 & 3 \\ -1 & 4-5 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so the 4-eigenspace is 1-dimensional and is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- Thus, the general solution to the system is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4x}}$.

- Example: Find all functions y_1, y_2, y_3 such that $\begin{aligned} y_1' &= y_1 - 3y_2 + 7y_3 \\ y_2' &= -y_1 - y_2 + y_3 \\ y_3' &= -y_1 + y_2 - 3y_3 \end{aligned}$.
  - The coefficient matrix is $A = \begin{bmatrix} 1 & -3 & 7 \\ -1 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}$, whose characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-1 & 3 & -7 \\ 1 & t+1 & -1 \\ 1 & -1 & t+3 \end{vmatrix} = t^3 + 3t^2 + 2t = t(t+1)(t+2)$.
  - Thus, the eigenvalues of A are $\lambda = 0, -1, -2$.
  - For $\lambda = 0$, we want to find the nullspace of $\begin{bmatrix} -1 & 3 & -7 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, so the 0-eigenspace is 1-dimensional and is spanned by $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.
  - For $\lambda = -1$, we want to find the nullspace of $\begin{bmatrix} -2 & 3 & -7 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, so the (-1)-eigenspace is 1-dimensional and is spanned by $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.
  - For $\lambda = -2$, we want to find the nullspace of $\begin{bmatrix} -3 & 3 & -7 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the (-2)-eigenspace is 1-dimensional and is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
  - Thus, the general solution to the system is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} e^{-x} + C_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-2x}}$.

- In the event that the coefficient matrix has complex-conjugate eigenvalues, we generally want to rewrite the resulting solutions as real-valued functions.
  - Suppose A has a complex eigenvalue $\lambda = a + bi$ with associated eigenvector $\mathbf{v} = \mathbf{w}_1 + i\mathbf{w}_2$. Then $\bar{\lambda} = a - bi$ has an eigenvector $\overline{\mathbf{v}} = \mathbf{w}_1 - i\mathbf{w}_2$ (the conjugate of $\mathbf{v}$), so we obtain the two solutions $e^{\lambda x}\mathbf{v}$ and $e^{\bar{\lambda} x}\overline{\mathbf{v}}$ to the system $\mathbf{y}' = A\mathbf{y}$.
  - Now we observe that $\frac{1}{2}(e^{\lambda x}\mathbf{v} + e^{\bar{\lambda} x}\overline{\mathbf{v}}) = e^{ax}(\mathbf{w}_1 \cos(bx) - \mathbf{w}_2 \sin(bx))$, and $\frac{1}{2i}(e^{\lambda x}\mathbf{v} - e^{\bar{\lambda} x}\overline{\mathbf{v}}) = e^{ax}(\mathbf{w}_1 \sin(bx) + \mathbf{w}_2 \cos(bx))$, and the latter solutions are real-valued.
  - Thus, to obtain real-valued solutions, we can replace the two complex-valued solutions $e^{\lambda x}\mathbf{v}$ and $e^{\bar{\lambda} x}\overline{\mathbf{v}}$ with the two real-valued solutions $e^{ax}(\mathbf{w}_1 \cos(bx) - \mathbf{w}_2 \sin(bx))$ and $e^{ax}(\mathbf{w}_1 \sin(bx) + \mathbf{w}_2 \cos(bx))$, which are simply the real part and imaginary part of $e^{\lambda x}\mathbf{v}$ respectively.

- Example: Find all real-valued functions y_1 and y_2 such that $\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}$.

- The coefficient matrix is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, whose characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t & -1 \\ 1 & t \end{vmatrix} = t^2 + 1$.
- Thus, the eigenvalues of A are $\lambda = \pm i$.
- For $\lambda = i$, we want to find the nullspace of $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$, so the i-eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1 \\ i \end{bmatrix}$.
- For $\lambda = -i$ we can take the complex conjugate of the eigenvector for $\lambda = i$ to see that $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector.
- The general solution, as a complex-valued function, is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{ix} + C_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-ix}$.
- We want real-valued solutions, so we must replace the complex-valued solutions $\begin{bmatrix} 1 \\ i \end{bmatrix} e^{ix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-ix}$ with real-valued ones.
- We have $\lambda = i$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i$ so that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Thus, the equivalent real-valued solutions are $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(x) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(x) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}$.
- The system's solution is then $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} \cos(x) \\ -\sin(x) \end{bmatrix} + C_2 \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}}$.

- Example: Find all real-valued functions y_1 and y_2 such that $\begin{array}{rcl} y_1' & = & 3y_1 - 2y_2 \\ y_2' & = & y_1 + y_2 \end{array}$.
  - The coefficient matrix is $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$, whose characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-3 & 2 \\ -1 & t-1 \end{vmatrix} = t^2 - 4t + 5$.
  - Thus, the eigenvalues of A are $\lambda = 2 \pm i$ by the quadratic formula.
  - For $\lambda = 2 + i$, we want to find the nullspace of $\begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$, so the i-eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.
  - For $\lambda = 2 - i$ we can take the complex conjugate of the eigenvector for $\lambda = 2 + i$ to see that $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is an eigenvector.
  - The general solution, as a complex-valued function, is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+i)x} + C_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-i)x}$.
  - We want real-valued solutions, so we must replace the complex-valued solutions $\begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+i)x}$ and $\begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-i)x}$ with real-valued ones.
  - We have $\lambda = 2 + i$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$ so that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
  - Thus, the real-valued solutions are $e^{2x} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(x) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(x) \right)$ and $e^{2x} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(x) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(x) \right)$.
  - The system's solution is then $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 e^{2x} \begin{bmatrix} \cos(x) - \sin(x) \\ \cos(x) \end{bmatrix} + C_2 e^{2x} \begin{bmatrix} \sin(x) + \cos(x) \\ \sin(x) \end{bmatrix}}$.

- <u>Example</u>: Find all real-valued functions y_1, y_2, y_3 such that $\begin{aligned} y_1' &= 3y_1 - 7y_2 - 3y_3 \\ y_2' &= y_1 - 4y_2 - 2y_3 \\ y_3' &= y_1 + 2y_2 + 2y_3 \end{aligned}$.

  - The coefficient matrix is $A = \begin{bmatrix} 3 & -7 & -3 \\ 1 & -4 & -2 \\ 1 & 2 & 2 \end{bmatrix}$, whose characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-3 & 7 & 3 \\ -1 & t+4 & 2 \\ -1 & -2 & t-2 \end{vmatrix} = (t+1)(t^2 - 2t + 2)$.
  - The eigenvalues of A are $\lambda = -1,\ 1 \pm i$ by the quadratic formula.
  - For $\lambda = -1$, we want to find the nullspace of $\begin{bmatrix} -4 & 7 & 3 \\ -1 & 3 & 2 \\ -1 & -2 & -3 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the (-1)-eigenspace is 1-dimensional and spanned by $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.
  - For $\lambda = 1+i$, we want to find the nullspace of $\begin{bmatrix} -2+i & 7 & 3 \\ -1 & 5+i & 2 \\ -1 & -2 & -1+i \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 5 & 0 & 1-3i \\ 0 & 5 & 2-i \\ 0 & 0 & 0 \end{bmatrix}$, so the $(1+i)$-eigenspace is spanned by $\begin{bmatrix} -1+3i \\ -2+i \\ 5 \end{bmatrix}$.
  - For $\lambda = 1-i$ we can take the complex conjugate of the eigenvector for $\lambda = 1+i$ to see that $\begin{bmatrix} -1-3i \\ -2-i \\ 5 \end{bmatrix}$ is an eigenvector.
  - The general solution, as a complex-valued function, is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-x} + C_2 \begin{bmatrix} -1+3i \\ -2+i \\ 5 \end{bmatrix} e^{-(1+i)} + C_3 \begin{bmatrix} -1-3i \\ -2-i \\ 5 \end{bmatrix} e^{-(1-i)}$, but we need to replace the complex-valued solutions with real-valued ones.
  - We have $\lambda = 1+i$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} i$ so that $\mathbf{w}_1 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.
  - Thus, the real-valued solutions are $e^x \left(\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \cos(x) - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \sin(x) \right)$ and $e^{2x} \left(\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \sin(x) + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \cos(x) \right)$.
  - Then $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-x} + C_2 e^x \begin{bmatrix} -\cos(x) - 3\sin(x) \\ -2\cos(x) - \sin(x) \\ 5\cos(x) \end{bmatrix} + C_3 e^x \begin{bmatrix} -\sin(x) + 3\cos(x) \\ -2\sin(x) + \cos(x) \\ 5\sin(x) \end{bmatrix}}$.

Well, you're at the end of my handout. Hope it was helpful.