Differential Equations (part 2): Linear Differential Equations (by Evan Dummit, 2016, v. 2.01)

Contents

5 Linear Differential Equations

In this chapter our goal is to study linear differential equations, and our treatment will be generally similar to our discussion of first-order equations. Solving general linear differential equations explicitly is typically very difficult, unless we are lucky and the equation has a particularly nice form: thus, we will instead first show some general facts about the structure of the solutions to a linear differential equation. We will then narrow our focus to a specific classes of linear differential equations whose solutions we can give an algorithm for finding: the linear differential equations with constant coefficients.

We will finish with a discussion of some of the applications of linear differential equations, which arise primarily in the context of physics.

5.1 General Linear Differential Equations

- The general nth-order linear differential equation can be written in the form $y^{(n)} + P_n(x)\,y^{(n-1)} + \cdots + P_2(x)\,y' + P_1(x)\,y = Q(x)$, for some functions $P_n(x), \cdots, P_2(x), P_1(x)$, and $Q(x)$.
  - Note that $y^{(n)}$ denotes the nth derivative of y.
- Our goal in this section is to study the general solution structure of this differential equation.
  - We can only give a method for writing down the full set of solutions for a small class of linear equations: namely, linear differential equations with constant coefficients.
  - There are a few equation types (e.g., Euler equations like $x^2y'' + xy' + y = 0$) which can be reduced to constant-coefficient equations via substitution. We will not discuss this, nor will we discuss nonlinear higher-order equations (whose solutions are in general even harder to find).

5.1.1 Terminology and Classification

- We first revisit some terminology for classifying different kinds of differential equations.
- Definition: An nth order differential equation is an equation in which the highest derivative is the nth derivative.
  - Example: The equations $y' + xy = 3x^2$ and $y' \cdot y = 2$ are first-order.
  - Example: The equation $y'' + y' + y = 0$ is second-order.
- Definition: A differential equation is linear if it is a linear combination of y and its derivatives. (Note that coefficients are allowed to be functions of x.) In other words, if there are no terms like y^2, or $(y')^3$, or $y \cdot y'$, or e^y.
  - Example: The equations $y' + xy = 3x^2$ and $y'' + y' + y = 0$ are linear.
  - Example: The equations $y' \cdot y = 3x^2$ and $y'' + e^y = 0$ are not linear.
- Definition: The standard form of a linear differential equation is when it is written with all terms involving y or higher derivatives on one side, and functions of the variable on the other side: that is, when it has the form $y^{(n)} + P_n(x)\, y^{(n-1)} + \cdots + P_2(x)\, y' + P_1(x)\, y = Q(x)$ for some functions $P_n(x), \cdots, P_2(x), P_1(x)$, and $Q(x)$.
  - Example: The equation $y'' + y' + y = 0$ is in standard form.
  - Example: The equation $y' = 3x^2 - xy$ is not in standard form.
- Definition: A linear differential equation $y^{(n)} + P_n(x)\, y^{(n-1)} + \cdots + P_2(x)\, y' + P_1(x)\, y = Q(x)$ in standard form is homogeneous if $Q(x)$ is the zero function, and it is nonhomogeneous otherwise. It has constant coefficients if the functions P_1, ... , P_n are all constants.
  - Example: The equation $y'' + y' + y = 0$ is homogeneous with constant coefficients.
  - Example: The equation $y''' + 3y' = e^x$ is nonhomogeneous with constant coefficients.
  - Example: The equation $y' + xy = 3x^2$ is nonhomogeneous.

5.1.2 Solution Structure and Existence-Uniqueness Theorem

- Like with first-order equations, we have an existence-uniqueness theorem for the general linear differential equation $y^{(n)} + P_n(x)\, y^{(n-1)} + \cdots + P_2(x)\, y' + P_1(x)\, y = Q(x)$.
  - Roughly speaking, in analogy with the first-order linear differential equation $y' + P(x)y = Q(x)$, where one integration is needed to solve the equation, we would need to integrate n times to find the solution to a general linear differential equation. This would introduce n arbitrary constants of integration, so we should expect there to be n parameters in the general solution.
  - If we also specify n initial conditions, then (in principle) we should expect there to be a unique solution, since the initial conditions would give n linear equations in the n parameters appearing in the general solution. Indeed, this is exactly what happens:
- Theorem (Existence-Uniqueness for Linear Equations): For any a, if $P_n(x), \cdots, P_1(x)$ and $Q(x)$ are functions continuous on an interval containing a, then there is a *unique* solution on that interval to the initial value problem $y^{(n)} + P_n(x)\, y^{(n-1)} + \cdots + P_2(x)\, y' + P_1(x)\, y = Q(x)$, for any initial conditions $y(a) = b_1$, $y'(a) = b_2$, $\cdots$, and $y^{(n-1)}(a) = b_n$.
  - As with the first-order existence-uniqueness theorem, the proof of this result is rather technically involved and we will omit it.
- The solutions to an nth order linear differential equation have a structure very similar to those of a system of linear equations in n variables.

- Proposition: Suppose y_{par} is one solution to the linear differential equation $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y=Q(x)$. Then the general solution y_{gen} to this equation may be written as $y_{gen}=y_{par}+y_{hom}$, where y_{hom} is a solution to the homogeneous equation $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y=0$.
  - Proof: Suppose that y_1 and y_2 are solutions to the general equation. We claim that their difference y_2-y_1 is a solution to the homogeneous equation: this implies the statement of the proposition.
  - To see this we subtract $(y_2-y_1)^{(n)}+\cdots+P_1(x)\,(y_2-y_1)=\left[y_2^{(n)}+\cdots+P_1(x)\,y_2\right]-\left[y_1^{(n)}+\cdots+P_1(x)\,y_1\right]=Q(x)-Q(x)=0$, where we used the property that the kth derivative of a difference is the difference of the kth derivatives.
  - Remark: Another way to interpret this result is to observe that the map L sending y to $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y$ is a linear transformation. Then $L(y_1)=Q(x)=L(y_2)$ implies $L(y_1-y_2)=Q(x)-Q(x)=0$ by properties of linear transformations, meaning that y_1-y_2 is a solution to the homogeneous equation.
- The solutions to a homogeneous linear differential equation have even more structure:
- Theorem (Homogeneous Linear Equations): If $P_n(x),\cdots,P_1(x)$ are continuous functions on an interval I, then the set of solutions to the homogeneous nth order equation $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y=0$ on I is an n-dimensional vector space.
  - Proof: First, we will check the subspace criterion for solutions to $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y=0$:
    - [S1]: The zero function is a solution.
    - [S2]: If y_1 and y_2 are solutions, then by adding the equations $y_1^{(n)}+P_n(x)\cdot y_1^{(n-1)}+\cdots+P_1(x)\cdot y_1=0$ and $y_2^{(n)}+P_n(x)\cdot y_2^{(n-1)}+\cdots+P_1(x)\cdot y_2=0$ and using properties of derivatives shows that $(y_1+y_2)^{(n)}+P_n(x)\cdot(y_1+y_2)^{(n-1)}+\cdots+P_1(x)\cdot(y_1+y_2)=0$, so y_1+y_2 is also a solution.
    - [S3]: If α is a scalar and y_1 is a solution, then scaling $y_1^{(n)}+P_n(x)\cdot y_1^{(n-1)}+\cdots+P_2(x)\cdot y_1'+P_1(x)\cdot y_1=0$ by α and using properties of derivatives shows that $(\alpha y_1)^{(n)}+P_n(x)\cdot(\alpha y_1)^{(n-1)}+\cdots+P_1(x)\cdot(\alpha y_1)=0$, so αy_1 is also a solution.
  - Now we need to show that the solution space is n-dimensional. We will do this by finding a basis.
    - Choose any a in I. By the existence part of the existence-uniqueness theorem, for each $1\le i\le n$ there exists a function y_i such that $y_i^{(i-1)}(a)=1$ and $y_i^{(k)}=0$ for all $0\le k\le n-1$ with $k\ne i$.
    - The functions $y_1,y_2,\ldots,y_n$ are linearly independent because their Wronskian matrix evaluated at $x=a$ is the identity matrix. (In particular, the Wronskian is not the zero function.)
    - Now suppose y is any solution to the homogeneous equation, with $y(a)=a_1$, $y'(a)=a_2$, ... , $y^{(n-1)}(a)=a_n$.
    - Then the function $z=a_1y_1+a_2y_2+\cdots+a_ny_n$ also has $z(a)=a_1$, $z'(a)=a_2$, ... , $z^{(n-1)}(a)=a_n$, and is a solution to the homogeneous equation.
    - But by the uniqueness part of the existence-uniqueness theorem, there is only one such function, so we must have $y(x)=z(x)$ for all x: therefore $y=a_1y_1+a_2y_2+\cdots+a_ny_n$, meaning that y is in the span of $y_1,y_2,\ldots,y_n$.
    - This is true for any solution function y, so $y_1,y_2,\ldots,y_n$ span the solution space. Since they are also linearly independent, they form a basis of the solution space, and because there are n of them, we see that the solution space is n-dimensional.
- Example: Solve the homogeneous equation $y''(x)=0$ and verify that the solution space is 2-dimensional.
  - We can just integrate twice to see that the solutions are $\boxed{y(x)=Ax+B}$, for arbitrary constants A and B.
  - Indeed, as the theorem dictates, the solution space is 2-dimensional, spanned by the two basis elements 1 and x.
- If we combine the above results, we can write down a fairly nice form for the solutions of a general linear differential equation:

- Corollary: The general solution to the nonhomogeneous equation $y^{(n)}+P_n(x)\,y^{(n-1)}+\cdots+P_2(x)\,y'+P_1(x)\,y = Q(x)$ has the form $y = y_p + C_1y_1 + C_2y_2 + \cdots + C_ny_n$, where y_p is any one particular solution of the nonhomogeneous equation, $y_1, \ldots, y_n$ are a basis for the solutions to the homogeneous equation, and $C_1, \ldots, C_n$ are arbitrary constants.
  - This corollary says that, in order to find the general solution, we only need to find one function which satisfies the nonhomogeneous equation, and then solve the homogeneous equation.
- Example: Find the general solution to the differential equation $y''(x) = e^x$.
  - We can just try simple functions until we discover that $y(x) = e^x$ has $y''(x) = e^x$, and so $y = e^x$ is a particular solution.
  - Then we need only solve the homogeneous equation $y''(x) = 0$, whose solutions we saw are $Ax + B$.
  - Thus the general solution to the general equation $y''(x) = e^x$ is $\boxed{y(x) = e^x + Ax + B}$.
  - We can also verify that if we impose the initial conditions $y(0) = c_1$ and $y'(0) = c_2$, then (as the existence-uniqueness theorem dictates) there is a unique solution $y = e^x + (c_2 - 1)x + (c_1 - 1)$.

5.2 Homogeneous Linear Equations with Constant Coefficients

- The general linear homogeneous differential equation with constant coefficients is $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$, where $a_{n-1}, \cdots, a_0$ are some constants.
  - From the existence-uniqueness theorem we know that the set of solutions is an n-dimensional vector space.
  - Based on solving first-order linear homogeneous equations (i.e., $y' + ky = 0$), we might expect the solutions to involve exponentials. If we try setting $y = e^{rx}$ then after some arithmetic we end up with $r^ne^{rx} + a_{n-1}r^{n-1}e^{rx} + \cdots + a_1re^{rx} + a_0e^{rx} = 0$. Multiplying both sides by e^{-rx} and cancelling yields the characteristic equation $\boxed{r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0}$.
  - If we can find n values of r satisfying this nth-degree polynomial (i.e., if we can factor the polynomial and see that it has n distinct roots) then the theorem tells us we will have found all of the solutions. If we are unlucky and the polynomial has a repeated root, then we need to try something else.
  - If there are non-real roots (note they will come in complex conjugate pairs) $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ then we would end up with e^{r_1x} and e^{r_2x} as our solutions. But we really want real-valued solutions, and e^{r_1x} and e^{r_2x} have complex numbers in the exponents. To fix this, we can just write out the real and imaginary parts using Euler's Theorem and take linear combinations to obtain the two real-valued solutions $e^{\alpha x}\sin(\beta x) = \dfrac{1}{2i}\left[e^{r_1x} - e^{r_2x}\right]$ and $e^{\alpha x}\cos(\beta x) = \dfrac{1}{2}\left[e^{r_1x} + e^{r_2x}\right]$.
  - Taking motivation from the case of $y^{(k)} = 0$, whose characteristic equation is $r^k = 0$ (with the k-fold repeated root 0) and whose solutions are $y(x) = A_1 + A_2x + A_3x^2 + \cdots + A_kx^{k-1}$, we can guess that if other roots are repeated, we want to multiply the corresponding exponentials e^{rx} by a power of x.
  - If we put all of these ideas together we can prove that this general outline will, in fact, give us n linearly independent functions, and hence gives the general solution to any homogeneous linear differential equation with constant coefficients.
- Theorem (Homogeneous Constant-Coefficient Equations): Suppose that $y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y = 0$ is a homogeneous linear equation with constant coefficients. If the associated characteristic equation $r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$ factors as $(r - r_1)^{s_1}(r - r_2)^{s_2}\cdots(r - r_k)^{s_k} = 0$ for complex numbers $r_1, \ldots, r_k$, then the functions $\{e^{r_1x},\, xe^{r_1x}, \ldots, x^{s_1-1}e^{r_1x}\}$, ... , $\{e^{r_kx},\, xe^{r_kx}, \ldots, x^{s_k-1}e^{r_kx}\}$ form a basis for the solution space of the homogeneous equation, where in the event that we have a pair of complex-conjugate roots $\alpha \pm \beta i$, the associated solutions $e^{(\alpha\pm\beta i)x}$ are replaced by $e^{\alpha x}\sin(\beta x)$ and $e^{\alpha x}\cos(\beta x)$.
  - Proof: For the statement about the complex-conjugate solutions, observe $e^{\alpha x}\sin(\beta x) = \dfrac{1}{2i}\left[e^{(\alpha+\beta i)x} - e^{(\alpha-\beta i)x}\right]$ and $e^{\alpha x}\cos(\beta x) = \dfrac{1}{2}\left[e^{(\alpha+\beta i)x} - e^{(\alpha-\beta i)x}\right]$, and conversely $e^{(\alpha+\beta i)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $e^{(\alpha+\beta i)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$, so we may freely convert between the complex exponentials and the trigonometric function solution forms. We will now work only with exponentials.

- Observe that there are n functions in the list $\{e^{r_1 x}, xe^{r_1 x}, \dots, x^{s_1-1}e^{r_1 x}\}$, ... , $\{e^{r_k x}, xe^{r_k x}, \dots, x^{s_k-1}e^{r_k x}\}$. It is a mostly straightforward (if tediously lengthy) calculation using the Wronskian that these functions are linearly independent, which we will omit.
- By the existence-uniqueness theorem, it is therefore sufficient to show that these n functions are all solutions to the homogeneous equation: we will then have n linearly independent functions in the n-dimensional solution space, so they necessarily form a basis.
- To show that these functions are actually solutions, we will use the language of linear transformations: first, let D represent the linear transformation sending a function to its derivative so that $Df = f'$.
- Now observe that $D^2 f = (f')' = f''$ is the second derivative, and by extension $D^n f = f^{(n)}$ is the nth derivative.
- More generally still, if we define the operator $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$, it is easy to verify that this is a linear transformation and that $Lf = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f$. In particular, the solutions to the original homogeneous equations are the functions y with $Ly = 0$ (i.e., the kernel of L).
- Next, notice that if p and q are any polynomials, then $p(D)q(D) = q(D)p(D)$: this follows by expanding out the products and applying the facts that $a_i D^i b_j D^j = (a_i b_j)D^{i+j} = b_j D^j a_i D^i$ for any constants a_i, b_j and any powers D^i, D^j.
- Repeatedly applying the above property, starting from the polynomial identity $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = (r-r_1)^{s_1}(r-r_2)^{s_2}\cdots(r-r_k)^{s_k}$, yields the equality $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 = (D-r_1)^{k_1}\cdots(D-r_j)^{k_j}$.
- Now we observe that $(D-r)x^n e^{rx} = r\,x^{n-1}e^{rx}$, and therefore $(D-r)^s x^d e^{rx} = 0$ whenever d is an integer with $0 \le d \le s-1$.
- At last we are finished: applying $L = (D-r_1)^{s_1}\cdots(D-r_k)^{s_k}$ to any function on the list $\{e^{r_i x}, xe^{r_i x}, \dots, x^{s_i-1}e^{r_i x}\}$ will yield zero, since the term $(D-r_i)^{s_i}$ applied to each of these functions is zero. Therefore, all the functions on our list are mapped to zero by L, meaning that they are all solutions of the homogeneous equation as claimed.

- To solve a linear homogeneous differential equation with constant coefficients, follow these steps:
  - Step 1: Rewrite the differential equation in the standard form $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ (if necessary).
  - Step 2: Factor the characteristic equation $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$.
  - Step 3: For each irreducible factor in the characteristic equation, write down the corresponding terms in the solution:
    * For terms $(r-\alpha)^s$ where is a real number, include terms of the form $e^{\alpha x}$, $xe^{\alpha x}, \cdots, x^{s-1}e^{\alpha x}$.
    * For terms $(r^2+cr+d)^s$ with roots $r = \alpha \pm \beta i$, include the terms $e^{\alpha x}\sin(\beta x)$, $xe^{\alpha x}\sin(\beta x), \cdots, x^{s-1}e^{\alpha x}\sin(\beta x)$ and $e^{\alpha x}\cos(\beta x)$, $xe^{\alpha x}\cos(\beta x), \cdots, x^{s-1}e^{\alpha x}\cos(\beta x)$.
  - Step 4: If given additional conditions on the solution, solve for the coefficients (if necessary).

- Example: Find all functions y such that $y'' + y' - 6 = 0$.
  - The characteristic equation is $r^2 + r - 6 = 0$ which has roots $r = 2$ and $r = -3$.
  - We have two distinct real roots, so our terms are e^{2x} and e^{-3x}. So the general solution is $\boxed{y = C_1 e^{2x} + C_2 e^{-3x}}$.

- Example: Find all functions y such that $y'' - 2y' + 1 = 0$, with $y(0) = 1$ and $y'(0) = 2$.
  - The characteristic equation is $r^2 - 2r + 1 = 0$ which has only the solution $r = 1$.
  - There is a double root at $r = 1$, so our terms are e^x and xe^x. Hence the general solution is $y = C_1 e^x + C_2 xe^x$.
  - Plugging in the two conditions gives $1 = C_1 \cdot e^0 + C_2 \cdot 0$, and $2 = C_1 e^0 + C_2\left[(0+1)e^0\right]$ from which $C_1 = 1$ and $C_2 = 1$. Hence the particular solution requested is $\boxed{y = e^x + xe^x}$.

- Example: Find all real-valued functions y such that $y'' = -4y$.
  - The standard form here is $y'' + 4y = 0$.
  - The characteristic equation is $r^2 + 4 = 0$ which has roots $r = 2i$ and $r = -2i$.
  - We have two complex-conjugate roots. Since the problem asks for real-valued functions we use the functions $\cos(2x)$ and $\sin(2x)$ in place of e^{2ix} and e^{-2ix} to obtain the general solution $\boxed{y = C_1 \cos(2x) + C_2 \sin(2x)}$.

- Example: Find all real-valued functions y such that $y^{(5)} + 5y^{(4)} + 10y''' + 10y'' + 5y' + y = 0$.
  - The characteristic equation is $r^5 + 5r^4 + 10r^3 + 10r^2 + 5r + 1 = 0$ which factors as $(r+1)^5 = 0$.
  - We have a 5-fold repeated root $r = -1$. Thus the terms are e^{-x}, xe^{-x}, x^2e^{-x}, x^3e^{-x}, and x^4e^{-x}. Hence the general solution is $\boxed{y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} + C_4x^3e^{-x} + C_5x^4e^{-x}}$.

- Example: Find all real-valued functions y such that $y''' + 2y'' + 5y' = 26y$.
  - The standard form here is $y''' + 2y'' + 5y' - 26y = 0$.
  - The characteristic equation is $r^3 + 2r^2 + 5r - 26 = 0$.
  - To solve this cubic we can search for small rational-root solutions, which must be divisors of the constant term -26. Trying the values ± 1, ± 2, ± 13, ± 26 reveals that $r = 2$ is a solution.
  - Factoring then yields $(r-2)(r^2 + 4r + 13) = 0$, and by the quadratic formula we see that the roots of the quadratic are $r = -2 \pm 3i$.
  - We have the real root 2 and two complex-conjugate roots $-2 \pm 3i$. Since the problem asks for real-valued functions we use the functions $e^{-2x}\cos(3x)$ and $e^{-2x}\sin(3x)$ in place of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ to get the general solution $\boxed{y = C_1e^{2x} + C_2e^{-2x}\cos(3x) + C_3e^{-2x}\sin(3x)}$.

- Example: Find all real-valued functions y whose fourth derivative is the same as y.
  - This is the equation $y'''' = y$, or in standard form, $y'''' - y = 0$.
  - The characteristic equation is $r^4 - 1 = 0$ which factors as $(r+1)(r-1)(r+i)(r-i) = 0$.
  - We have the four roots $1, -1, i, -i$. Thus the terms are e^x, e^{-x}, $\cos(x)$, and $\sin(x)$. Hence the general solution is $\boxed{y = C_1e^x + C_2e^{-x} + C_3\cos(x) + C_4\sin(x)}$.

5.3 Non-Homogeneous Linear Equations with Constant Coefficients

- The general linear differential equation with constant coefficients is of the form $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = Q(x)$, where $a_{n-1}, \cdots, a_0$ are some constants and $Q(x)$ is some function of x.

- From the general theory, all we need to do is find one solution to the general equation, and find all solutions to the homogeneous equation. Since we know how to solve the homogeneous equation in full generality, we just need to develop some techniques for finding one solution to the general equation.

- There are essentially two ways of doing this.
  - The method of undetermined coefficients is just a fancy way of making an an educated guess about what the form of the solution will be and then checking if it works. It will work whenever the function $Q(x)$ is a linear combination of terms of the form $x^ke^{\alpha x}$ (where k is an integer and α is a complex number): thus, for example, we could use the method for something like $Q(x) = x^3e^{8x}\cos(x) - 4\sin(x) + x^{10}$ but not something like $Q(x) = \tan(x)$.
  - Variation of parameters is a more complicated method which uses some linear algebra and cleverness to use the solutions of the homogeneous equation to find a solution to the non-homogeneous equation. It will always work, for any function $Q(x)$, but generally requires more setup and computation.

5.3.1 The Method of Undetermined Coefficients, Annihilators

- The idea behind the method of undetermined coefficients is that we can guess what our solution should look like (up to some coefficients we have to solve for), if the nonhomogeneous portion $Q(x)$ involves sums and products of polynomials, exponentials, and trigonometric functions.
  - Specifically, we try a solution $y =$ [stuff], where the "stuff" is a sum of things similar to the terms in $Q(x)$.
  - The method of undetermined coefficients is essentially educated guessing: we guess the general form of a particular solution based on the form of $Q(x)$, and then plug in to check
- Here is the method of undetermined coefficients:
  - Step 1: Generate the "first guess" for the trial solution as follows:
    - Replace all numerical coefficients of terms in $Q(x)$ with variable coefficients. If there is a sine (or cosine) term, add in the companion cosine (or sine) terms, if they are missing. Then group terms of $Q(x)$ into "blocks" of terms which are the same up to a power of x, and add in any missing lower-degree terms in each "block".
    - Thus, if a term of the form $x^n e^{rx}$ appears in $Q(x)$, fill in the terms of the form $e^{rx} \cdot [A_0 + A_1 x + \cdots + A_n x^n]$, and if a term of the form $x^n e^{\alpha x} \sin(\beta x)$ or $x^n e^{\alpha x} \cos(\beta x)$ appears in $Q(x)$, fill in the terms of the form $e^{\alpha x} \cos(\beta x) \cdot [D_0 + D_1 x + \cdots + D_n x^n] + e^{\alpha x} \sin(\beta x) [E_0 + E_1 x + \cdots + E_n x^n]$.
  - Step 2: Solve the homogeneous equation, and write down the general solution.
  - Step 3: Compare the "first guess" for the trial solution with the solutions to the homogeneous equation. If any terms overlap, multiply all terms in the overlapping "block" by the appropriate power of x which will remove the duplication.
  - Step 4: If asked to find a solution to the differential equation, plug in the trial solution into the equation and solve for the coefficients.
- Here is a series of examples demonstrating the procedure for generating the trial solution:
  - Example: $y'' - y = x$.
    - Step 1: We fill in the missing constant term in $Q(x)$ to get $D_0 + D_1 x$.
    - Step 2: The general homogeneous solution is $A_1 e^x + A_2 e^{-x}$.
    - Step 3: There is no overlap, so the trial solution is $\boxed{D_0 + D_1 x}$.
  - Example: $y'' + y' = x - 2$.
    - Step 1: We have $D_0 + D_1 x$.
    - Step 2: The general homogeneous solution is $A + Be^{-x}$.
    - Step 3: There is an overlap (the solution D_0) so we multiply the corresponding trial solution terms by x, to get $D_0 x + D_1 x^2$. Now there is no overlap, so $\boxed{D_0 x + D_1 x^2}$ is the trial solution.
  - Example: $y'' - y = e^x$.
    - Step 1: We have $D_0 e^x$.
    - Step 2: The general homogeneous solution is $Ae^x + Be^{-x}$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by x, to get $D_0 x e^x$. Now there is no overlap, so $\boxed{D_0 x e^x}$ is the trial solution.
  - Example: $y'' - 2y' + y = 3e^x$.
    - Step 1: We have $D_0 e^x$.
    - Step 2: The general homogeneous solution is $Ae^x + Bxe^x$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by x^2, to get rid of the overlap, giving us the trial solution $\boxed{D_0 x^2 e^x}$.
  - Example: $y'' - 2y' + y = x^3 e^x$.

* Step 1: We fill in the lower-degree terms to get $D_0 e^x + D_1 x e^x + D_2 x^2 e^x + D_3 x^3 e^x$.
* Step 2: The general homogeneous solution is $A_0 e^x + A_1 x e^x$.
* Step 3: There is an overlap (namely $D_0 e^x + D_1 x e^x$) so we multiply the trial solution terms by x^2 to get $\boxed{D_0 x^2 e^x + D_1 x^3 e^x + D_2 x^4 e^x + D_3 x^5 e^x}$ as the trial solution.

- Example: $y'' + y = \sin(x)$.
  * Step 1: We fill in the missing cosine term to get $D_0 \cos(x) + E_0 \sin(x)$.
  * Step 2: The general homogeneous solution is $A\cos(x) + B\sin(x)$.
  * Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms by x to get $D_0 x\cos(x) + E_0 x\sin(x)$. There is now no overlap so $\boxed{D_0 x\cos(x) + E_0 x\sin(x)}$ is the trial solution.

- Example: $y'' + y = x\sin(x)$.
  * Step 1: We fill in the missing cosine term and then all the lower-degree terms to get $D_0 \cos(x) + E_0 \sin(x) + D_1 x\cos(x) + E_1 x\sin(x)$.
  * Step 2: The general homogeneous solution is $A\cos(x) + B\sin(x)$.
  * Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms in that group by x to get $\boxed{D_0 x\cos(x) + E_0 x\sin(x) + D_1 x^2 \cos(x) + E_1 x^2 \sin(x)}$, which is the trial solution since now there is no overlap.

- Example: $y''' - y'' = x + xe^x$.
  * Step 1: We fill in the lower-degree term for xe^x and the lower-degree term for x, to get $A_0 + A_1 x + B_0 e^x + B_1 x e^x$.
  * Step 2: The general homogeneous solution is $C_0 + C_1 x + De^x$.
  * Step 3: There are overlaps in both groups of terms: $A_0 + A_1 x$ and $B_0 e^x$ each overlap, so we multiply the "x" group by x^2 and the "e^x" group by x to get rid of the overlaps. There are now no additional overlapping terms, so the trial solution is $\boxed{A_0 x^2 + A_1 x^3 + B_0 x e^x + B_1 x^2 e^x}$.

- Example: $y'''' + 2y'' + y = xe^x + x\cos(x)$.
  * Step 1: We fill in the lower-degree term for xe^x, then the missing sine term for $x\cos(x)$, and then the lower-degree terms for $x\cos(x)$ and $x\sin(x)$, to get $A_0 e^x + A_1 x e^x + D_0 \cos(x) + E_0 \sin(x) + D_1 x\cos(x) + E_1 x\sin(x)$.
  * Step 2: The general homogeneous solution is $B_0 \cos(x) + C_0 \sin(x) + B_1 x\cos(x) + C_1 x\sin(x)$.
  * Step 3: There is an overlap (namely, all of $D_0 \cos(x) + E_0 \sin(x) + D_1 x\cos(x) + D_1 x\sin(x)$) so we multiply that group by x^2 to get rid of the overlap. There are no additional overlapping terms, so the trial solution is $\boxed{A_0 e^x + A_1 x e^x + D_0 x^2 \cos(x) + E_0 x^2 \sin(x) + D_1 x^3 \cos(x) + E_1 x^3 \sin(x)}$.

• Here is a series of examples finding the general trial solution and then solving for the coefficients:

- Example: Find a function y such that $y'' + y' + y = x$.
  * The procedure produces our trial solution as $y = D_0 + D_1 x$, because there is no overlap with the solutions to the homogeneous equation.
  * If $y = D_0 + D_1 x$ then $y' = D_1$ and $y'' = 0$ so plugging in yields $y'' + y' + y = 0 + (D_1) + (D_1 x + D_0) = x$, so that $D_1 = 1$ and $D_0 = -1$.
  * So our solution is $\boxed{y = x - 1}$.

- Example: Find a function y such that $y'' - y = 2e^x$.
  * The procedure gives the trial solution as $y = D_0 x e^x$, since $D_0 e^x$ overlaps with the solution to the homogeneous equation.
  * If $y = D_0 x e^x$ then $y'' = D_0(x+2)e^x$ so plugging in yields $y'' - y = [D_0(x+2)e^x] - [D_1 x\, e^x] = 2e^x$. Solving yields $D_0 = 1$, so our solution is $\boxed{y = xe^x}$.

- Example: Find a function y such that $y'' - 2y' + y = x + \sin(x)$.

* The procedure gives the trial solution as $y = (D_0 + D_1 x) + (D_2 \cos(x) + D_3 \sin(x))$, by filling in the missing constant term and cosine term, and because there is no overlap with the solutions to the homogeneous equation.
* Then we have $y'' = -D_2 \cos(x) - D_3 \sin(x)$ and $y' = D_1 - D_2 \sin(x) + D_3 \cos(x)$ so plugging in yields

$$y'' - 2y' + y = [-D_2 \cos(x) - D_3 \sin(x)] - 2\,[D_1 - D_2 \sin(x) + D_3 \cos(x)] + [D_0 + D_1 x + D_2 \cos(x) + D_3 \sin(x)]$$

and setting this equal to $x + \sin(x)$ then requires $D_0 - 2D_1 = 0$, $D_1 = 1$, $D_2 + 2D_3 - D_2 = 1$, $D_3 - 2D_2 - D_3 = 0$, so our solution is $\boxed{y = x + 2 + \frac{1}{2}\cos(x)}$.

- Example: Find all functions y such that $y'' + y = \sin(x)$.
  * The solutions to the homogeneous system $y'' + y = 0$ are $y = C_1 \cos(x) + C_2 \sin(x)$.
  * Then the procedure gives the trial solution for the non-homogeneous equation as $y = D_0 x \cos(x) + D_1 x \sin(x)$, by filling in the missing cosine term and then multiplying both by x due to the overlap with the solutions to the homogeneous equation.
  * We can compute (eventually) that $y'' = -D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)$.
  * Plugging in yields $y'' + y = (-D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)) + (D_0 x \sin(x) + D_1 x \cos(x))$, and so setting this equal to $\sin(x)$, we obtain $D_0 = 0$ and $D_1 = -\frac{1}{2}$.
  * Therefore the set of solutions is $\boxed{y = -\frac{1}{2} x \cos(x) + C_1 \cos(x) + C_2 \sin(x)}$, for constants C_1 and C_2.

- The formal idea behind the method of undetermined coefficients is that nonhomogeneous terms given by powers of x times exponentials, sines, or cosines are all sent to zero ("annihilated") by some polynomial in the differential operator D.
  - Recall that the differential operator D is the linear transformation that maps a function to its derivative: thus, for example, $D(x^2) = 2x$ and $D(e^x) = e^x$.
  - If we apply that polynomial in D which sends $Q(x)$ to zero to both sides of the original equation, we will end up with a homogeneous equation whose characteristic polynomial is the product of the original characteristic polynomial and the characteristic polynomial for $Q(x)$.
  - We can then solve this homogeneous equation using our earlier techniques to write down its general solution. However, only some of the solution functions will satisfy the original equation: we then need to plug in to determine the ones that work.
  - This procedure is sometimes called the "method of annihilators", although it is really the same thing as the method of undetermined coefficients.

- Example: Find the general solution to $y'' - y = x^2$ using the method of annihilators.
  - If we differentiate both sides 3 times (i.e., apply the differential operator D^3 to both sides), in order to get rid of the x^2 term on the right-hand side then we get $y^{(5)} - y^{(3)} = 0$, which has characteristic polynomial $r^5 - r^3 = (r^2 - 1) \cdot (r^3)$.
  - Observe that this polynomial is the product of the characteristic polynomial $r^2 - 1$ of the original equation and the polynomial r^3 corresponding to D^3.
  - Now $y^{(5)} - y^{(3)} = 0$ is homogeneous, so we can write down the general solution to obtain $y = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-x}$.
  - Now we plug in: $y'' = 2C_3 + C_4 e^x + C_5 e^{-x}$ so $y'' - y = -C_3 x^2 - C_2 x + (2C_3 - C_1)$. Setting the coefficients equal gives $C_3 = -1$, $C_2 = 0$, $C_1 = -2$, so the general solution is $y = \boxed{-x^2 - 2 + C_4 e^{-x} + C_5 e^x}$.

- Example: Find the form of a solution to $y'' + y = x + \sin(x)$ using the method of annihilators.
  - This is the same as the equation $(D^2 + 1) \cdot y = x + \sin(x)$.
  - We want to apply the operator D^2 to get rid of the x term, and the operator $D^2 + 1$ to get rid of the $\sin(x)$ term.

- The new differential equation, after we are done, is $(D^2+1)(D^2)(D^2+1)\cdot y = 0$.
- The characteristic polynomial is $(r^2+1)^2 r^2$, which has a double root at each of $r=0$ and $\pm i$.
- The general solution to the equation is thus $y = \boxed{C_1\sin(x) + C_2 x\sin(x) + C_3\cos(x) + C_4 x\cos(x) + C_5 + C_6 x}$.

5.3.2 Variation of Parameters

- The method of undetermined coefficients does not work for all possible nonhomogeneous linear differential equations with constant coefficients: it only works when the nonhomogeneous term $Q(x)$ is a sum of products of polynomials, exponentials, sines, and cosines.
  - It cannot, for example, solve the differential equation $y'' + y = \sec(x)$.
- There is a more robust method known as variation of parameters which can solve any nonhomogeneous linear differential equation provided the solutions to the homogeneous equation are known. Here is the derivation of the method:
  - Suppose $y_1, \cdots, y_n$ are the n linearly independent solutions to the homogeneous equation $y^{(n)} + P_n(x)\,y^{(n-1)} + \cdots + P_2(x)\,y' + P_1(x)\,y = 0$.
  - We will look for functions $v_1, \cdots, v_n$ making $y_p = v_1\cdot y_1 + v_2\cdot y_2 + \cdots + v_n\cdot y_n$ a solution to the nonhomogeneous equation $y^{(n)} + P_n(x)\,y^{(n-1)} + \cdots + P_2(x)\,y' + P_1(x)\,y = Q(x)$.
  - We do this by requiring $v_1', v_2', \cdots, v_n'$ to satisfy the system of equations

$$\begin{aligned}
v_1'\cdot y_1 + v_2'\cdot y_2 + \cdots + v_n'\cdot y_n &= 0\\
v_1'\cdot y_1' + v_2'\cdot y_2' + \cdots + v_n'\cdot y_n' &= 0\\
\vdots \quad &\vdots \quad \vdots\\
v_1'\cdot y_1^{(n-2)} + v_2'\cdot y_2^{(n-2)} + \cdots + v_n'\cdot y_n^{(n-2)} &= 0\\
v_1'\cdot y_1^{(n-1)} + v_2'\cdot y_2^{(n-1)} + \cdots + v_n'\cdot y_n^{(n-1)} &= Q(x).
\end{aligned}$$

  - We choose these relations so that the kth derivative $y_p^{(k)}$ is equal to $v_1\cdot y_1^{(k)} + v_2\cdot y_2^{(k)} + \cdots + v_n\cdot y_n^{(k)}$ for $0 \le k \le n-1$, and so that $y_p^{(n)} = Q(x) + v_1\cdot y_1^{(n)} + v_2\cdot y_2^{(n)} + \cdots + v_n\cdot y_n^{(n)}$. (It is straightforward to verify these statements by differentiating the defining expressing for y_p the appropriate number of times.)
  - We can then do some algebra to see that $y_p^{(n)} + P_n(x)\,y_p^{(n-1)} + \cdots + P_2(x)\,y_p' + P_1(x)\,y_p = Q(x)$. (When we add everything up we will get terms of the form $v_i(y_i^{(n)} + P_n(x)\,y_i^{(n-1)} + \cdots + P_2(x)\,y_i' + P_1(x)\,y_i)$ which are zero because y_i is a solution to the homogeneous equation.)
  - So all we need to do is solve the system of n linear equations for $v_1', \ldots, v_n'$ above: we can either do this directly using row-reduction, or we can use Cramer's rule to see that $v_i' = \dfrac{W_i(x)}{W(x)}$ where W is the Wronskian of the functions $y_1, y_2, \cdots, y_n$ and W_i is the same Wronskian determinant except with the ith column replaced with the column vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ Q(x) \end{bmatrix}$.
  - We can then integrate to find each v_i and thus obtain the solution to the system.
- Here is the general procedure for using variation of parameters to solve a nonhomogeneous equation $y_p^{(n)} + P_n(x)\,y_p^{(n-1)} + \cdots + P_2(x)\,y_p' + P_1(x)\,y_p = Q(x)$:
  - Step 1: Solve the corresponding homogeneous equation $y^{(n)} + P_n(x)\,y^{(n-1)} + \cdots + P_2(x)\,y' + P_1(x)\,y = 0$ and find n (linearly independent) solutions $y_1, \cdots, y_n$.

- Step 2: Find functions $v_1, \cdots, v_n$ making $y_p = v_1 \cdot y_1 + v_2 \cdot y_2 + \cdots + v_n \cdot y_n$ a solution to the original equation: the desired functions satisfy $\boxed{v_i' = \dfrac{W_i(x)}{W(x)}}$, where W is the Wronskian of the functions $y_1, y_2, \cdots, y_n$ and W_i is the same Wronskian determinant except with the ith column replaced with $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ Q(x) \end{bmatrix}$.
- Step 3: Write down the particular solution to the nonhomogeneous equation, $\boxed{y_p = v_1 \cdot y_1 + v_2 \cdot y_2 + \cdots + v_n \cdot y_n}$.
- Step 4: If asked, add the particular solution to the general solution to the homogeneous equation, to find all solutions of the nonhomogeneous equation. This will yield $\boxed{y = y_p + C_1 y_1 + \cdots + C_n y_n}$. Plug in any extra conditions given to solve for coefficients.

- Example: Find all functions y for which $y'' + y = \sec(x)$ using variation of parameters.
  - The homogeneous equation is $y'' + y = 0$ which has two independent solutions $y_1 = \cos(x)$ and $y_2 = \sin(x)$.
  - Since $Q(x) = \sec(x)$, we can now compute the necessary determinants: $W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1$, $W_1 = \begin{vmatrix} 0 & \sin(x) \\ \sec(x) & \cos(x) \end{vmatrix} = -\sin(x) \cdot \sec(x)$, and $W_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec(x) \end{vmatrix} = \cos(x) \cdot \sec(x) = 1$.
  - Plugging in to the formulas gives $v_1' = \dfrac{W_1}{W} = -\sin(x) \cdot \sec(x) = -\tan(x)$ and $v_2' = \dfrac{W_2}{W} = \cos(x) \cdot \sec(x) = 1$.
  - Integrating the relations yields $v_1 = \ln(\cos(x))$ and $v_2 = x$, so we obtain the particular solution $y_p = \ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)$.
  - The general solution is therefore $\boxed{y = [\ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)] + C_1 \sin(x) + C_2 \cos(x)}$.

- Example: Find all functions y for which $y''' - y'' + y' - y = e^x$ using variation of parameters.
  - The homogeneous equation is $y''' - y'' + y' - y = 0$ which has characteristic polynomial $r^3 - r^2 + r - 1 = (r-1)(r^2+1)$, so we can take the three independent solutions of the homogeneous equation as $y_1 = \cos(x)$, $y_2 = \sin(x)$, and $y_3 = e^x$.
  - Since $Q(x) = e^x$, we can now compute the determinants $W = \begin{vmatrix} \cos(x) & \sin(x) & e^x \\ -\sin(x) & \cos(x) & e^x \\ -\cos(x) & -\sin(x) & e^x \end{vmatrix} = 2e^x$, $W_1 = \begin{vmatrix} 0 & \sin(x) & e^x \\ 0 & \cos(x) & e^x \\ e^x & -\sin(x) & e^x \end{vmatrix} = e^{2x}(\sin(x) - \cos(x))$, $W_2 = \begin{vmatrix} \cos(x) & 0 & e^x \\ -\sin(x) & 0 & e^x \\ -\cos(x) & e^x & e^x \end{vmatrix} = -e^{2x}(\cos(x) + \sin(x))$, and finally $W_3 = \begin{vmatrix} \cos(x) & \sin(x) & 0 \\ -\sin(x) & \cos(x) & 0 \\ -\cos(x) & -\sin(x) & e^x \end{vmatrix} = e^x$.
  - Plugging in to the formulas gives $v_1' = \dfrac{1}{2}e^x(\sin(x) - \cos(x))$, $v_2' = -\dfrac{1}{2}e^x(\cos(x) + \sin(x))$, and $v_3' = \dfrac{1}{2}$.
  - Integrating yields $v_1 = -\dfrac{1}{2}e^x \cos(x)$, $v_2 = -\dfrac{1}{2}e^x \sin(x)$, and $v_3 = \dfrac{1}{2}x$, so we obtain the particular solution $y_p = -\dfrac{1}{2}e^x \cos^2(x) - \dfrac{1}{2}e^x \sin^2(x) + \dfrac{1}{2}xe^x = -\dfrac{1}{2}e^x + \dfrac{1}{2}xe^x$.
  - The general solution is therefore $\boxed{y = \dfrac{1}{2}xe^x + C_1 \cos(x) + C_2 \sin(x) + C_3 e^x}$. (Note that we absorbed the $-\dfrac{1}{2}e^x$ term from the particular solution into $C_3 e^x$.)
  - Note that we could also have used the method of undetermined coefficients for this problem: it will, of course, give the same answer.

5.4 Applications of Second-Order Equations

- In this section, we will discuss applications of second-order differential equations to physics.
  - In kinematics, the key ingredient is Newton's second law, which states that $F = ma$, where F is the total sum of forces acting on an object whose mass is m and whose acceleration is a.
  - By writing down all the forces acting on an object and relating them to the object's position and velocity, we obtain a differential equation whose solution will characterize the object's motion.
  - If the forces involved in the problem depend on an object's position, then the resulting differential equation is second-order, since acceleration is the second derivative of position.
- The prototypical examples of this type are problems involving a spring, because Hooke's law dictates that the restoring force of the spring is proportional to the displacement from equilibrium.
  - Specifically, if a spring's displacement from equilibrium is x, then the restoring force is given by $F_{\text{spring}} = -kx$ for a constant k known as the "spring constant" (for that particular spring). The units of k are force per distance.
  - The basic setup of a typical spring problem is as follows: an object of mass m kg is attached to one end of a spring with spring constant k N/m whose other end is fixed. The mass is displaced some distance d m from the equilibrium position, and we wish to find the object's position $x(t)$ as a function of time.
  - We also assume that the object is sliding along a surface with constant damping coefficient μ, meaning that it applies a damping force $F_{\text{friction}} = -\mu v$ where $v = x'$ is the velocity of the object.
  - Finally, we also include the possibility that a motor or other device imparts some additional nonconstant force $F = Q(t)$ (possibly varying with time) to the object.
  - By Newton's second law, we therefore obtain the constant-coefficient second-order linear differential equation $mx'' = -kx - \mu x' + Q(t)$ for x, which we can rewrite in the more standard form $mx'' + \mu x' + kx = Q(t)$.
- In circuit analysis, the key ingredient is Kirchhoff's second law, which states that the total sum of the voltage drops around any closed circuit is zero. In order to use this law, one needs to be given the values of voltage drops across circuit components.
  - Recall that $q(t)$ denotes electrical charge measured in coulombs (C), and $i(t) = dq/dt$ denotes electrical current measured in amperes (A).
  - A resistor will resist the flow of charge through it in direct proportion to the current. Specifically, by Ohm's law, the voltage drop across a resistor is $\Delta V_R = Ri$ where R is the resistance in ohms (Ω).
  - A capacitor will store charge and resist the passage of current in direct proportion to the amount of electrical charge. The voltage drop across a capacitor is $\Delta V_C = \frac{1}{C}q$ where C is the capacitance of the capacitor in farads (F).
  - An inductor will resist a change in the electrical current in direct proportion to the rate of change of the current. The voltage drop across an inductor is $\Delta V_L = L\frac{di}{dt}$, where L is the inductance of the inductor in henrys (H).
  - A voltage source (such as a battery creating a direct current, or an alternator creating an alternating current) will produce an electromotive force. The voltage drop across the source is equal to $\Delta V_E = -E(t)$ for some function $E(t)$ in volts (V).
  - Thus, in a simple RLC circuit, containing a resistor, capacitor, inductor, and a voltage source connected in series in a circle, applying Kirchhoff's law immediately yields the relation $L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$.
  - If we write everything in terms of q, we get a constant-coefficient second-order linear differential equation for q, namely $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$.
- We therefore see that, for both of these problems, the resulting differential equation is a constant-coefficient second-order linear differential equation for the quantity we are interested in (either the position x or the charge q), and so the two problems are, in fact, essentially equivalent.
- We will now focus only on studying spring problems, since their behavior is easier to visualize intuitively.

5.4.1 Mechanical Oscillations

- Let us first treat the case of a spring-mass system with no damping:
- Example: An object of mass m kg is attached to a spring of spring constant k N/m whose other end is fixed. The object is displaced a distance d m from the equilibrium position of the spring and is let go with velocity v_0 m/s towards equilibrium at time $t = 0$. If the object is restricted to sliding horizontally on a frictionless surface, find its position as a function of time.
  - Since there is no damping, from our analysis above the differential equation is $mx'' + kx = 0$, where x denotes the object's position relative to equilibrium.
  - We also have the initial conditions $x(0) = d$ and $x'(0) = -v_0$: since positive displacement is in the direction opposite equilibrium, the fact that the initial velocity is toward equilibrium indicates that $x'(0)$ is negative.
  - We can then rewrite the differential equation in standard form $x'' + \frac{k}{m}x = 0$.
  - The characteristic equation is then $r^2 + \frac{k}{m} = 0$ with roots $r = \pm\sqrt{\frac{k}{m}}i$, so the general solution is $\boxed{x = C_1\cos(\omega t) + C_2\sin(\omega t)}$, where $\omega = \sqrt{\frac{k}{m}}$.
  - The initial conditions give $d = x(0) = C_1$ and $-v_0 = x'(0) = \omega C_2$, so $C_1 = d$ and $C_2 = -v_0/\omega$. Thus, the solution we want is $\boxed{x = d \cdot \cos(\omega t) - \frac{v_0}{\omega} \cdot \sin(\omega t)}$ where $\omega = \sqrt{\frac{k}{m}}$.
  - Note that the solution we have obtained makes sense in the context of this problem, since on a frictionless surface we should expect that the object's motion would be purely oscillatory: it should just bounce back and forth along the spring forever since there is nothing to slow its motion.
  - We can even see that the form of the solution agrees with our intuition: the fact that the frequency $\omega = \sqrt{\frac{k}{m}}$ increases with bigger spring constant but decreases with bigger mass makes sense, as a stronger spring (with larger k) should pull back harder on the object and cause it to oscillate more quickly, while a heavier object should resist the spring's force and oscillate more slowly.
- Now let us examine the more general case of a mass attached to a spring sliding on a surface with friction, but no external driving force:
- Example: An object of mass m kg is attached to a spring of spring constant k N/m whose other end is fixed. The object is displaced from equilibrium and then released. If the object is restricted to sliding horizontally on surface with damping coefficient μ, describe its position qualitatively as a function of time.
  - If the position is $x(t)$, our earlier analysis with Newton's second law yields the differential equation $mx'' + \mu x' + kx = 0$.
  - The characteristic equation is then $mr^2 + \mu r + k = 0$, whose roots are $r = -\frac{\mu}{2m} \pm \frac{\sqrt{\mu^2 - 4mk}}{2m}$.
  - Depending on the sign of the quantity $\mu^2 - 4mk$ under the square root, the values of r will take different forms and give different types of solutions.
  - Overdamped Case: If $\mu^2 - 4mk > 0$, the quadratic has two real roots (both negative) and we end up with general solutions that are the sum of two exponentially-decaying functions: specifically, $x = C_1e^{-r_1t} + C_2e^{-r_2t}$ where $r_1 = \frac{\mu}{2m} + \frac{\sqrt{\mu^2 - 4mk}}{2m}$ and $r_2 = \frac{\mu}{2m} - \frac{\sqrt{\mu^2 - 4mk}}{2m}$ (note that r_1 and r_2 are both positive). Physically, as we can see from the condition $\mu^2 - 4mk > 0$, this means we have "too much" damping, since we can see from the form of the solution function that the position of the object will just slide back towards its equilibrium at $x = 0$ without oscillating at all. This is known as the "overdamped" case.

- Critically Damped Case: If $\mu^2 - 4mk = 0$, the quadratic has a double root, and so the general solutions have the form $(C_1 + C_2 t)e^{-rt}$ for a positive real number r, which when graphed is a slightly-slower-decaying exponential function that still does not oscillate, but could possibly cross the position $x = 0$ once depending on the values of C_1 and C_2. This is known as the "critically damped" case.
- Underdamped Case: If $\mu^2 - 4mk < 0$ and $\mu > 0$, the quadratic has two complex-conjugate roots and we end up with general solutions of the form $e^{-\alpha t}[C_1 \cos(\omega t) + C_2 \sin(\omega t)]$, where $\alpha = \dfrac{\mu}{2m}$ and $\omega = \dfrac{\sqrt{4mk - \mu^2}}{2m}$. When graphed, this is a sine curve times an exponentially-decaying function. Physically, the position of the object will still tend toward $x = 0$, but the sine and cosine terms will ensure that it continues oscillating: this means that there is some damping but not enough to eliminate the oscillations entirely. This is known as the "underdamped" case.
- Undamped Case: If $\mu = 0$, we saw earlier that the solutions are of the form $y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$ where $\omega^2 = k/m$. Since there is no damping, this is referred to as the "undamped" case.

- If we have particular numbers we can write down the solutions more explicitly.
- Example: A 1-kilogram mass is attached to one end of a spring with spring constant 5N/m whose other end is fixed. The mass is displaced 2 meters and then released from rest. If the object is restricted to sliding horizontally on surface with damping coefficient $\mu = 2$, find the mass's position at time t seconds.
  - From our analysis, we obtain the differential equation $x'' + 2x' + 5x = 0$, with initial condition $x(0) = 2$ meters and $x'(0) = 0$.
  - The characteristic equation is $r^2 + 2r + 5 = 0$, whose roots are $r = -1 \pm 2i$.
  - Thus, the general solution of the equation is $x(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$.
  - Some arithmetic then gives $x(0) = C_1$ and $x'(0) = 2C_2 - C_1$, so the initial conditions give $C_1 = 2$ and $C_2 = 1$.
  - Thus, the position at time t is $x(t) = \boxed{2e^{-t}\cos(2t) + e^{-t}\sin(2t)}$.

5.4.2 Resonance and Forcing

- We can also study what happens in a spring-mass problem when we introduce an external driving force. It is possible to write down the general solution to the equation $mx'' + \mu x' + kx = A\cos(\alpha t + \varphi)$ for an arbitrary sinusoidal driving force $A\cos(\alpha t + \varphi)$, but it is somewhat painful to analyze the solutions in full generality. We will instead treat a few basic examples that give the general flavor of things:
- Example: An object of mass m is attached to a spring of spring constant k whose other end is fixed, and is sliding on a frictionless surface at a frequency $\omega = \sqrt{k/m}$. Examine what happens to the object's motion if an external driving force $Q(t) = A\cos(\omega t)$ is applied which oscillates at the same frequency ω.
  - By our analysis above, the differential equation is $m \cdot x'' + k \cdot x = Q(t)$, where $\omega = \sqrt{k/m}$.
  - If we divide through by m and put in $k = m \cdot \omega^2$ we get the simpler equation $x'' + \omega^2 x = \dfrac{A}{m}\cos(\omega t)$.
  - Now we use the method of undetermined coefficients to find a solution to this differential equation.
  - We would like to try something of the form $x = D_1 \cos(\omega t) + D_2 \sin(\omega t)$, but this will not work because functions of that form are already solutions to the homogeneous equation $x'' + \omega^2 x = 0$.
  - Instead the method instructs that the appropriate solution will be of the form $x = D_1 t \cdot \cos(\omega t) + D_2 t \cdot \sin(\omega t)$. We can use a trigonometric formula (the sum-to-product formula) to rewrite this as $x = \boxed{Dt \cdot \cos(\omega t + \phi)}$, where ϕ is an appropriate "phase shift". (We can solve for the coefficients in terms of A, m, ω but it will not be so useful.)
  - We can see from this formula that as t grows, so does the "amplitude" Dt: in other words, as time goes on, the object will continue oscillating with frequency ω around its equilibrium point, but the swings back and forth will get larger and larger.

- The phenomenon in the example above is called <u>resonance</u>: applying a driving force to a system at its natural oscillation frequency will create a resonant effect that increases the amplitude in an unbounded manner.
  - One can observe this phenomenon by sitting in a rocking chair or by swinging an object back and forth: some experimentation will quickly reveal that the most effective way to rock the chair or swing the object is to push back and forth at the same frequency at which the object is already moving.

- <u>Example</u>: Repeat the above analysis with an external force $Q(t) = A\cos(\omega_1 t)$ oscillating at a frequency $\omega_1 \neq \omega$.
  - In this case (using the same argument as above) we have $x'' + \omega^2 x = \dfrac{A}{m}\cos(\omega_1 t)$.
  - The trial solution (again by undetermined coefficients) is $x(t) = B\cos(\omega_1 t)$, where $B = \dfrac{A/m}{\omega^2 - \omega_1^2}$.
  - Thus the overall general solution is $x = \boxed{\dfrac{A/m}{\omega^2 - \omega_1^2}\cos(\omega_1 t) + C_1\cos(\omega t) + C_2\sin(\omega t)}$ for some constants C_1 and C_2 that depend on the initial conditions.
  - As we can see, if ω_1 and ω are far apart (i.e., the driving force is oscillating at a very different frequency from the frequency of the original system) then B will be small, and so the overall change $B\cos(\omega_1 t)$ that the driving force adds will be relatively small.
  - However, if ω_1 and ω are very close to one another (i.e., the driving force is oscillating at a frequency close to that of the original system) then B will be large, and so the driving force will cause the system to oscillate with a much bigger amplitude.
  - As ω_1 approaches ω, the amplitude $B = \dfrac{A/m}{\omega^2 - \omega_1^2}$ will go to ∞, which agrees with the behavior seen in the previous example where ω_1 actually equals ω.

- Understanding how resonance arises (and how to minimize it!) is a very, very important application of differential equations to structural engineering.
  - A poor understanding of resonance is something which has caused a number of structural disasters: ultimately, resonance caused the Broughton bridge to fall down in 1831, the Tacoma Narrows bridge to collapse in 1940, Partnair Flight 394 to crash in 1989, and Seoul's Techno-Mart mall to be evacuated in 2011. The particular culprits in those respective disasters were soldiers marching in lockstep, wind vibrations, improperly calibrated aircraft parts, and an enthusiastic exercise class.
  - As illustrated in the examples above, resonance arises when an external force acts on a system at (or very close to) one of the system's "natural resonance frequencies".
  - Of course, resonance is not always bad. The idea of using the physical construction of an object to magnify a small oscillation into a large one is the principle behind the construction of most musical instruments, and is often also cited as the reason many people like to sing in the shower.

Well, you're at the end of my handout. Hope it was helpful.